The Maurer-Cartan form for Matrix groups

A write-up inspired by questions Junwen Liao asked about p 153 to 154 of Flanders book .

Let G be a Lie group which arises as a closed subgroup of the general linear $Gl(N,\mathbb{R})$. Some families of examples of such G's are the classical Lie groups, the special linear group $Sl(N)$, the orthogonal group $O(N)$, special orthogonal group, also called the rotation group $SO(N)$, the unitary group $U(n)$, and the symplectic group $Sp(n)$. The HW 6 is about (most) of these.

Write $\mathbb{M}_N$ for the vector space of all N by N matrices, so that

$$G \subset Gl(N) \subset \mathbb{M}_N$$

and $Gl(N)$ is an open set of $\mathbb{M}_N$, being the subset where the determinant is nonzero.

Taking a bit more global perspective on what Flanders wrote, we view the inclusion of G into $\mathbb{M}_N$ as a smooth map

$$X : G \to \mathbb{M}_N; X(h) = h.$$

into a vector space. Then X is a vector-valued map on G. Just like real-valued functions, vector valued maps admit differentials:

$$dX : TG \to \mathbb{M}_N \qquad , dX_h : T_hG \to T_h\mathbb{M}_N = \mathbb{M}_N \text{ for } h \in G, dX_h(v) = v$$

and the image of dX_h is the tangent space $T_hG \subset \mathbb{M}_N$, included as a subspace of $\mathbb{M}_N$. Alternatively, we can think of dX as a vector-valued one-form.

Remark: More generally if $X : Q \to \mathbf{V}$ is an imbedding of a manifold into a real vector space $\mathbf{V}$ then we can think of dX, the differential of the imbedding, as a vector-valued one-form on Q. This perspective is used in the theory of curves and of surfaces. See for example the final chapter of GP, the undergraduate differential geometry books by Singer-Thorpe or O'neill, and anything on surface theory by E. Cartan.

Here is the Maurer-Cartan form as Flanders writes it.

$$\omega_{Fl} = X^{-1}dX. \tag{1}$$

Exercise 1. *Embed S^1 into $\mathbb{M}_2$ in the standard way*

$$X(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Show that $\omega_{Fl} = Jd\theta$ where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

The one-form Flanders writes down is a special case of the Maurer-Cartan form as it is defined for any Lie group, matrix or not. Write

$$\mathfrak{g} = T_eG.$$

This fixed vector space $\mathfrak{g}$ is the Lie algebra of G. Write

$$L_g : G \to G; L_g(h) = gh$$

for the diffeomorphism of left translation of G by some fixed $g \in G$. (Its inverse is $L_{g^{-1}}$.) Then

$$(dL_g)_e : \mathfrak{g} \to T_gG$$

so that if we define

$$\omega(g) = (dL_g)_e^{-1} : T_gG \to \mathfrak{g}$$

then ω is defined for all $g \in G$ and varies smoothly with g. It is a $\mathfrak{g}$-valued one-form on G.

Definition 1. *The one-form just defined is the (left) Maurer-Cartan form.*

Proposition 1. *The form Flanders writes, (eq (1)) is the same as the Maurer-Cartan form defined above.*

I will prove the proposition below. Before I do let me quote Junwen. "In conclusion: The main part I am confused about is how to think of X in Flanders. Do I consider it as a point in G(a specific matrix) or an embedding or more strictly, as the inclusion map. "

The short answer to Junwen's question is " think of X both ways!"

A) For considerations of rigor think of X as the inclusion map

B) in order to do calculations quickly and intuitively, think of X as a variable point on X - which is to say, a matrix-valued function on G.

It is worth spending some time pondering these things. Confusion is natural in this world. Just keep working with it. After thinking about it awhile, you may start to see that A) and B) are really the same thing.

Proof of Proposition.

We have

$$L_g X = gX.$$

We can either restrict X to varying over G, or we can extend L_g to all of $\mathbb{M}_N$ so that X varies over $\mathbb{M}_N$. Since $g(X_1 + cX_2) = gX_1 + cgX_2$, viewed in the later way L_g becomes a linear operator on $\mathbb{M}_N$. Then the derivative of L_g is again L_g! - linear operators are the best linear approximations to themselves. Thus

$$(dL_g)(V) = gV$$

and this is true, regardless of whether we insist $V \in T_h G$ or let V be a random matrix in $\mathbb{M}_N$. Since $L_g^{-1} = L_{g^{-1}}$ we have that $(dL_g)_e^{-1}(v) = g^{-1}v$ and, upon restriction, this becomes a linear operator from $T_g G$ onto $\mathfrak{g} = T_e G$. Take $g \in G$ and $v \in T_g G$ arbitrary. Then:

$$\omega_{Fl}(g)(v) = X(g)^{-1} dX_g(v) = g^{-1}v = (dL_g)_e^{-1}(v)$$

proving that $\omega_{Fl} = \omega$. QED

Along the way of the proof, we also proved a

Key Fact If $G \subset \mathbb{M}_N$ is a matrix group and if $g \in G$ is fixed and L_g is restricted to the matrix group G, then the differential $(dL_g)_h$ at any point $h \in G$ is the linear map $T_h G \to T_{gh} G$ which sends v to gv. It is a linear isomorphism between subspaces of $\mathbb{M}_N$.

Junwen also asked: how do you show that the Maurer-Cartan form is left-invariant? (Trying to follow Flanders derivation of this fact spurred Junwen's question.)

Proposition 2. *The Maurer-Cartan form is left-invariant: For all $g \in G$ we have $L_g^*\omega = \omega$*

Proof 1. "By the book"

$$\begin{aligned}
(L_g^*\omega)(h)(X) &= \omega(L_g h)(dL_g X) &(2)\\
&= \omega(gh)(gX) &(3)\\
&= L_{(gh)^{-1}}(gX) &(4)\\
&= (gh)^{-1}(gX) &(5)\\
&= h^{-1}g^{-1}(gX) &(6)\\
&= h^{-1}(X) &(7)
\end{aligned}$$

but

$$\omega(h)(X) = L_{h^{-1}}X = h^{-1}X$$

So $L_A^*\omega = \omega$

Proof 2. By substitution.

Replace the variable $X \in G$ by $\tilde{X} = gX$ with g a constant matrrix in G. Then

$$\begin{aligned}
\tilde{X}^{-1}d(\tilde{X}) &= (Xg)^{-1}d(Xg) &(8)\\
&= (X^{-1}g^{-1})gdX &(9)\\
&= g^{-1}dg &(10)
\end{aligned}$$

so left-translation does not change the form of ω and hence leaves it invariant.

Summary. The Maurer-Cartan form ω is a smoothly varying collection of linear maps

$$\omega(g) : T_g G \to \mathfrak{g}.$$

so, a Lie-algebra valued one-form on the manifold G. For each fixed g in a matrix group it is the linear isomorphism given by

$$\omega(g) : v \mapsto g^{-1}v, v \in T_g G$$

Conclusion. The tangent space of a Lie group is trivial, or "globally parallelizable" Namely,the map which takes $(g, v) \in T_g G \to (g, \omega(g)v) \in G \times \mathfrak{g}$ defines a fiber-linear diffeomorphism $TG \to G \times \mathfrak{g}$. It turns the tangent bundle of G into the trivial vector bundle of the right dimension! This map is known as "left-trivialization".

Remark There is also a "right-trivialization" defined by the right-invariant Maurer-Cartan form $(dX)X^{-1}$.

Cor. The only possible Lie groups that are also spheres are S^0, S^1, S^3 and S^7. Of these, all but S^7 are Lie groups.

Proof. Homotopical and characteristic class arguments show that all the other spheres S^k, $k \neq 0, 1, 3, 7$ cannot be parallelized.

The Maurer-Cartan form may look kind of silly to you. After all , for matrix groups we showed that
$$\omega(g)(v) = g^{-1}v; v \in T_g G.$$
It becomes non-trivial when we start parameterizing G.

Exercise 2. *"Euler angles" are coordinates for the Lie group $SO(3)$ found by Euler. They send angles (θ, ϕ, ψ) to the 3 by 3 rotation matrix $X(\theta, \phi, \psi)$ which is given by*

$$\begin{pmatrix} \cos(\phi)\cos(\theta)\cos(\psi) - \sin(\phi)\sin(\psi) & \sin(\phi)\cos(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) & -\sin(\theta)\cos(\psi) \\ -\cos(\phi)\cos(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) & -\sin(\phi)\cos(\theta)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\theta)\sin(\psi) \\ \cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\theta) \end{pmatrix}$$

[I took this expression from p 10 of Whittaker's 'analytical Dynamics" text. You might look elsewhere for it and check and look up where they come from and what they mean.] Compute the Maurer-Cartan matrix $X^{-1}dX$ in terms of Euler angles, expanding the result one-form out in terms of the standard basis for $so(3)$. which consists of three skew-symmetric matrices having all but two entries 0, those nonzero entries being 1 and -1.

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