For next time.

A) Volunteer presentat...

$\Lambda^* V, \Lambda^* V^*$.

B) any of the H/W.

C) $\mathbb{C}P^1 = S^2$. = Riem sphere

$= \mathbb{C} \cup \{\infty\}$

$= \mathbb{R}^2 \cup \{\infty\}$

forms on $\mathbb{R}^n$.

- Grassman alg.
- Exterior alg.

1^{st} exterior algebra.

an $\mathbb{R}$ - algebra.

Basis $1, dx^1, \ldots, dx^n$.

Relations $dx^i \wedge dx^j = -dx^j \wedge dx^i$

Exer $(dx^1 \wedge dx^2 + dx^3 \wedge dx^4)^2 = 2 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$!

(Fl. writes : $dx^i dx^j = -dx^j dx^i$ etc.

Notation $\Lambda^* \mathbb{R}^n$, better: $\Lambda^*(\mathbb{R}^{n*})$

K-forms ~~at a~~ pt : $\Lambda^K \mathbb{R}^n$.

sums of order K monomials in the dx^i.

E. Cartan

Interpretation $\Lambda^K \mathbb{R}^{n*} \ni \omega$

= completely antisym. K-linear fns

$$\omega : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{K} \longrightarrow \mathbb{R}.$$

$$\Lambda^* \mathbb{R}^{n*} = \bigoplus_{k=0} \Lambda^k(\mathbb{R}^{n*})$$

$$\omega(av+w, v_2, \ldots, v_k)$$ (multi lin.)

$$= a\,\omega(v, v_2, \ldots, v_k) + \omega(w, v_2, \ldots, v_k)$$

$\forall v, w, v_i \in \mathbb{R}^n$. $\forall a \in \mathbb{R}$.

$$\omega(v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots v_k)$$

$$= -\omega(v_1, v_2, v_j, v_i, \ldots v_k)$$

eg: dx^i dual to $e_i = \frac{\partial}{\partial x^i} = (0 \ldots \underset{\substack{\uparrow \\ i\text{th}}}{1} \ldots 0)$

~~$dx^i \wedge dx^j$~~ $\quad dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$

so:

$$dx^i(v) = v^i \quad \text{if}$$

$$v = \sum v^j \frac{\partial}{\partial x^j}$$

$$(dx^i \wedge dx^j)(v,w) = dx^i(v)\,dx^j(w) - dx^i(w)\,dx^j(v)$$

$$= v^i w^j - w^j v^i$$

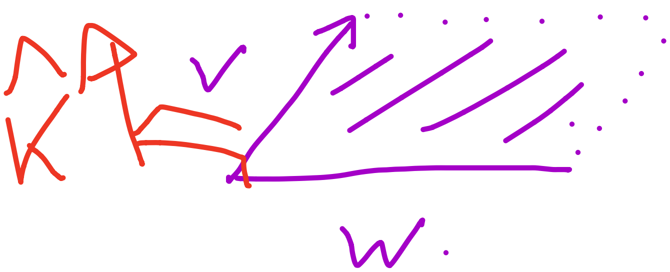
= signed area of proj of v, w onto $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}$ plane.

so on $\mathbb{R}^2$:

$dx^1 \wedge dx^2 \longleftrightarrow$ "det"

on $\mathbb{R}^3$

$dx^1 \wedge dx^2 \wedge dx^3 \longleftrightarrow$ "det" etc.



$\exists$ area

$= v^1 w^2 - v^2 w^1$

$$\|v \times w\| = \text{area}$$

$$v \times w = (\quad)\hat{k} \;;\; \|v \times w\| = |(\quad)|$$

"$\hat{k}$"

$$v \wedge w = (v_1 w_2 - v_2 w_1)\, e_1 \wedge e_2$$

$$v = v_1 e_1 + v_2 e_2$$

$$w = w_1 e_1 + w_2 e_2$$

stopped here.

Jan 5, 2021.

Then a k-form on $\mathbb{R}^n$ is a smooth map

$$\mathbb{R}^n \longrightarrow \Lambda^k \mathbb{R}^n.$$

in coord

1-form: $\sum a_i(x)\,dx^i$

$x = (x^1, \ldots, x^n)$

2-form $\sum a_{ij}(x)\,dx^i \wedge dx^j$

sum: either $i < j$

or: $a_{ij} = -a_{ji}$ ← Cartan prefers

3-form $\sum a_{ijk}(x)\,dx^i \wedge dx^j \wedge dx^k$

either $a_{ijk} = a_{kij} = a_{jki}$
$= -a_{jik} \ldots$

or insist
$i < j < k$

a $\frac{1}{3!}$ relating the 2-ways

On an manifold,

coord x^i

same deal.

locally $\sum a_I \, dx^I$

globally?

~~x~~ $u^i = u^i(x^1, \dots, x^n)$

inverse

$x^i = x^i(u^1, \dots, u^n)$.

$$dx^i = \sum \frac{\partial x^i}{\partial u^j} du^j$$

&

$$du^j = \sum \frac{\partial u^j}{\partial x^i} dx^i$$

induce change of coord in the form & show how to express them in another ~~basis~~ coord system.

Vectors:

$$v \in T_p M,$$

$$v = \sum v^i \frac{\partial}{\partial x^i} = \sum \tilde{v}^j \frac{\partial}{\partial u^j}$$

$$= \sum v^i \frac{\partial u^j}{\partial x^i} \frac{\partial}{\partial u^j}$$

$$\Rightarrow \boxed{\tilde{v}^j = \sum \frac{\partial u^j}{\partial x^i} v^i}$$

$$\begin{tikzcd} & W, U \subset M \arrow[dl, "x"'] \arrow[dr] & \\ \mathbb{R}^n \arrow[rr] & & \mathbb{R}^n \end{tikzcd}$$

$$u = u(x),$$

$$= \phi_u \circ \phi_x^{-1}(x)$$

$$= \psi(x)$$

$$\tilde{v} = d\psi_x v. \quad \text{rep.}$$ "Contravariant"

Forms. Exer show if

$$p = \sum p_i \, dx^i = \sum \tilde{p}_j \, du^j$$

Then $\tilde{p} = (d\psi_x^{-1})^+ p.$

"covariant"

verify it.

$$\sum p_i\, dx^i = \sum p_i \frac{\partial x^i}{\partial u^j}\, du^j$$

$$= \sum \tilde{p}_i\, du^i$$

so

$$\tilde{p}_i = \sum p_i \frac{\partial x^i}{\partial u^j}$$

But:

$$\left(\frac{\partial u^j}{\partial x^i}\right)^{-1} = \frac{\partial x^i}{\partial u^j}$$

& look at sum: now on top index not bottom