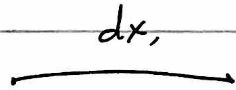
1B
?

k-form on a manifold:

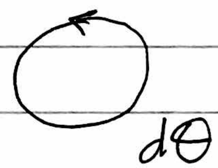
$$\omega \in \Gamma(M, \Lambda^k T^* M) := \Omega^k(M).$$

means in any set of loc. coord. ω of prev. form.

Egs $d=1$,



$M = [0,1]$



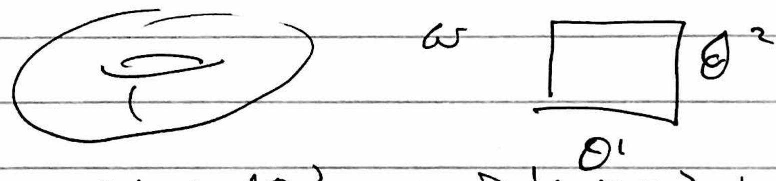
$M = S^1$

N.B. $\oint d\theta = 2\pi$

~~but dθ~~ so $d\theta \neq df'$

θ not a function!
rather "multi valued" mod $2\pi\mathbb{Z}$.

$d=2$



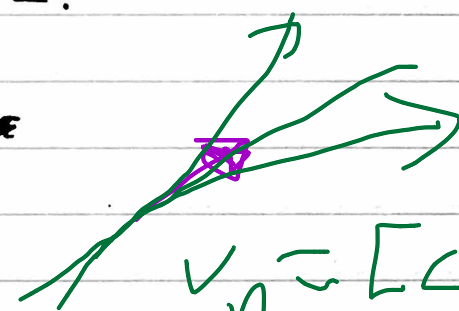
$a\,d\theta^1 + b\,d\theta^2, \in \Omega^1(\mathbb{T}^2)$
$d\theta^1 \wedge d\theta^2 \in \Omega^2(\mathbb{T}^2)$

Intrinsic Defn. of vectors ~~&~~ covectors
vector fields, one-forms on M.

vectors, ~~fields~~ $v \in T_pM$. , Space T_pM.

3 intrinsic defn.

a) derivations
b) curves
c) coordinates



$v_p = [c]$
$= \dot{c}(0)$

What are they?

0 -
1 - $c : (-\varepsilon, \varepsilon) \to M$.
b - $c(0) = p$, $c \sim \tilde{c}$ iff
$\tilde{c}(0) = p$. agree to 1st order

Dually for the space of covectors $\alpha \in T^*_pM$.

b'): via function $f : M \to \mathbb{R}$.

Suppose $f(p) = 0$.
Call two such fns f, g "equivalent" if $f - g$ vanishes to 1st order at p. ~~i.e.~~ :

Exer: ~~vanishe~~
if ψ vanishes to 1^{st} order in one coord system if vanishes to 1^{st} order in every coord system.

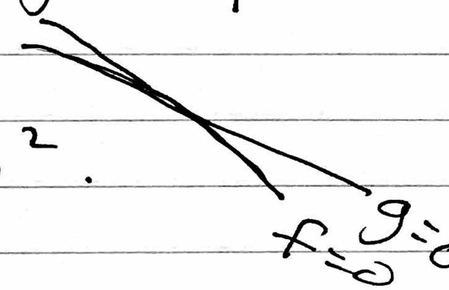
ideal in $C^\infty(M)$

Set $m_p = \{f \in C^\infty(M) : f(p) = 0\}$

ψ vanishes to 1^{st} order
$\iff \psi \in m_p^2 = \{\varphi = \sum c_i f_i g_i$
$f_i, g_i \in m_p\}$.

Thus, 2^{nd} def $\iff$

$$f \sim g \iff f - g \in m_p^2.$$



so:

$$T^*_p M = m_p / m_p^2.$$

N.B.:
$$C^\infty(M) \xrightarrow{\delta_p} \mathbb{R}$$
$$f \longmapsto f(p)$$

Ker δ_p $= m_p$ ✓

is an $\mathbb{R}$-alg homomorphism.
$$1 \longmapsto 1.$$

$\Rightarrow m_p$ is a max. ideal in the ring $C^\infty(M)$

get $\mathrm{der}_p = (m_p/m_p^2)^* = T_p$

$[f] = df_p \qquad m_p/m_p^2 = T_p^* M.$

works well in alg. geometry.

Duality: for $\quad v[f] = df_p(v_p) \in \mathbb{R}$

or: $\quad = \dfrac{d}{dt}\Big|_{t=0} f(c(t)),$

where $\dot{c}(0) = v_p$.

& $df_p = [f - f(p)] \in m_p/m_p^2$.

<u>v-fld:s</u> Derivations
$$C^\infty(M) \to C^\infty(M)$$

$$X[f + bg] = X[f] + b\,X[g]$$
$$X[fg] = f\,X[g] + g\,X[f]$$

one-forms: .. ?
well:
$$\Gamma(TM) \longrightarrow C^\infty(M)$$
by $\theta(X)(p) = \theta_p(X_p)$.

$$\theta(fX) = f\,\theta(X)$$

Straightening lemma:
v-flds: If $X(p) \neq 0$
$\exists$ coord centered at p
$x^1, \ldots, x^n$
s.t. $X = \frac{\partial}{\partial x^1}$ in a nbhd of p.

~~For forms. No! $D(p$~~
so: All non-vanishing v-flds
locally diffeomorphic.

For one-forms no:

$\exists\ \theta, \tilde{\theta}$ $\quad \theta(p) = \tilde{\theta}(p)$

yet $\nexists$ local diffeo F, $F^*\tilde{\theta} = \theta$.

why? d comes in

If assume $d\theta = 0$, $d\tilde{\theta} = 0$

& $\theta(p) \neq 0$, $\tilde{\theta}(p) \neq 0$

then $\exists$ diffeo.

Indeed lemma Say $\theta \in \Omega^1(M)$

& $\theta(p) \neq 0$ & near p

$d\theta = 0$.

Then $\exists$ loc. coord $x^1, \dots, x^n$

s.t.

$\theta = dx^1$ near p.