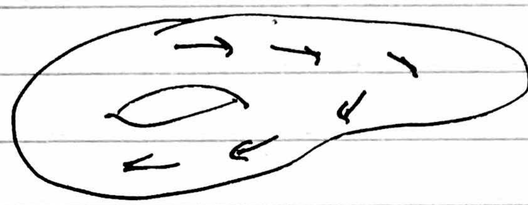
Vector field on M.

$X : M \longrightarrow TM$, smooth

$X(p) \in T_p M \quad \forall p \in M$.

picture



loc coord rep.

$X = \sum X^i(x)\, \partial_i$,

$\partial_i = \partial / \partial x^i$

One form :

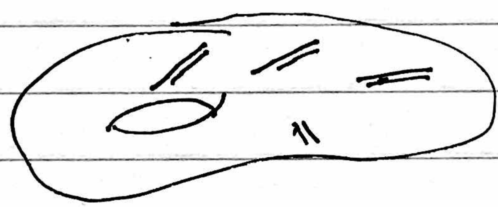
$\alpha : M \longrightarrow T^* M$ smooth

$\forall p \in M \quad \alpha(p) \in T^*_p M = (T_p M)^*$

Thus: $\forall\, \alpha(p) : T_p M \longrightarrow \mathbb{R}$.

loc coord rep. $\quad \alpha = \sum \alpha_i(x)\, dx^i$

pic



ω just

(α up to pos scale)

Ifn & Bo : $\alpha(X)(p) = \alpha(p)(X(p)) \in \mathbb{R}$.

Pairing: $\mathfrak{X}(M) \times \Omega^1(M) \longrightarrow \mathbb{R}$.

$\Gamma(TM)$

$$\begin{tikzcd} E = \bigcup_{p\in M} E_p \arrow[d] \\ M \arrow[u, bend left, "s"] \end{tikzcd}$$

$$s \in \Gamma(E)$$

$$s(p) \in E_p .$$

$$\Omega^1(M) = \Gamma(T^*M)$$

$$\Gamma(E) = \Gamma(M, E)$$

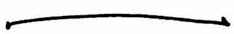
1B

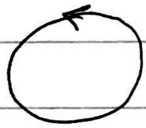
?

k-form on a manifold:

$$\omega \in \Gamma(M, \Lambda^k T^* M) := \Omega^k(M).$$

<u>means</u> in any set of loc. coord. ω of prev. form.

<u>Egs</u> $d=1$, dx,  $M = [0,1]$

 $M = S^1$

$d\theta$, $g(\theta)\,d\theta$

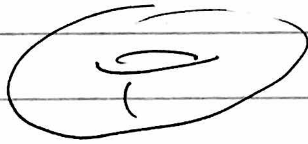
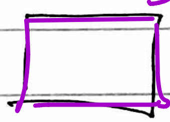
N.B. $\oint d\theta = 2\pi$

~~but $d\theta$~~ so $d\theta \neq df$

θ not a function!

rather "multi valued" mod $2\pi\mathbb{Z}$.

$M = \mathbb{R}$; $\Omega^1(\mathbb{R}) \simeq \{g(x)\,dx,\ g \text{ smooth}\}$

$d=2$  ω  θ^2

θ^1

$a\,d\theta^1 + b\,d\theta^2, \in \Omega^1(\mathbb{T}^2)$

$d\theta^1 \wedge d\theta^2 \in \Omega^2(\mathbb{T}^2)$

$$\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2.$$

K-form on M.

$$\omega : M \longrightarrow \Lambda^K T^*M.$$

$\forall p \quad \omega(p) \in \Lambda^K T^*_p M.$

coord rep. $\omega = \sum \omega_I(x) dx^I$

$I = (i_1, \dots, i_k)$; $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

$i_1 < i_2 < \dots < i_k$ or ω_I "skew"

Pulling back forms.

$m \in M \xrightarrow{F} N$ smooth map.

$\alpha \in \Omega^1(N)$ one form on N.

$(F^*\alpha)(m)$ by: $\in \Lambda^1 T^*M$

$$(F^*\alpha)(m)(v_m) = \alpha(F(m))(dF_m v_m)$$

(recall:

$v_m \qquad dF_m \qquad dF_m v_m$

$$T_m M \longrightarrow T_{F(m)} N$$

linear.

$\Lambda^k T^*N$

$F^*\alpha \uparrow \qquad F \qquad \alpha \uparrow$

$$M \longrightarrow N$$

$$\Omega(N) \xrightarrow{F^*} \Omega(M)$$

Pulling back k forms; similar.

$M \xrightarrow{F} N$; β k-form on N.

$F^*\beta$ a k-form on M.

$(F^*\beta)(m)(v_1, \dots, v_k)$; $v_i \in T_m M$.

$= \beta(F(m))(dF_m v_1, \dots, dF_m v_k)$.

abstraction

$(\Omega(M), d)$, $(\Omega(N), d)$ are

"~~ext~~ differential algebras"

F^* is an homomorphism

$(\Omega(N), d) \xrightarrow[F^*]{} (\Omega(M), d)$

meaning ...

Differential Graded Algebra.

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy$$

Exer 1) $F^* d = d F^*$.

2) $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$.

3) F^* is $\mathbb{R}$-linear

4) $(F \circ G)^* = G^* \circ F^*$; $M \xrightarrow{G} N \xrightarrow{F} X$

Example: Lets use to compute $F^*\omega$ for $\omega = \sum x^i dx^i$ on $\mathbb{R}^n$.

&

$$F(x) = Kx$$

$$\begin{aligned} F^*\left(\sum x^i dx^i\right) &= \sum F^*(x^i dx^i) \\ &= \sum (F^* x^i) F^* dx^i \\ &= \sum (F^* x^i)\, d(F^* x^i) \\ &= \sum K x^i d(K x^i) \\ &= K^2 \sum x^i dx^i \end{aligned}$$

N.B: $\Omega^0(M) \xrightarrow{F} \Omega^0(N)$

$$F^* f = f \circ F$$

so $(F^* \underset{\substack{\uparrow \\ f}}{x^i}) =$ ith component of $F(x)$ $= Kx^i$.

<u>Moral</u>: Pull back acts like substitution of variables.

Exercise.

1. $F: M \longrightarrow N$

$F(m) = n_0 = \text{const. point.}$

Show $F^*\omega = 0$ i.e. $\forall F^* = 0$.

for any $\omega \in \Omega^k(N)$.

2. $(x, y) \xrightarrow{\psi} (r, \theta)$

Compute

$\psi^* dr = \ldots x, y, dx, dy$

$\psi^* d\theta$