$\partial_i = \frac{\partial}{\partial x^i}$ $dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$

1

Conventions on $\wedge$, $\otimes$ etc.

Is $(dx^1 \wedge dx^2)(\partial_1, \partial_2) = 1$, or $\frac{1}{2}$?

Guillemin - Pollack: $\frac{1}{2}$.
Abraham-Marsden, Morita, Bourbaki, Lee, ...: 1
Flanders: non-committal.
Sjamaar?- non-committal? : E.C. find Sj's conventions, if any.
We will stick w "1". Then
if $\alpha_1, \ldots, \alpha_K \in V^*$
$v_1, \ldots, v_K \in V$

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_K)(v_1, \ldots, v_K) = \det(\alpha_i(v_j))$$

cf G.P, 4.2: exer 3; 10; {$= \frac{1}{K!}$ "

eg. $K = 2$.
$$(\alpha_1 \wedge \alpha_2)(v_1, v_2) = \alpha_1(v_1)\alpha_2(v_2) - \alpha_1(v_2)\alpha_2(v_1)$$

fn $\Rightarrow$ rep. signed vol. of parellipiped

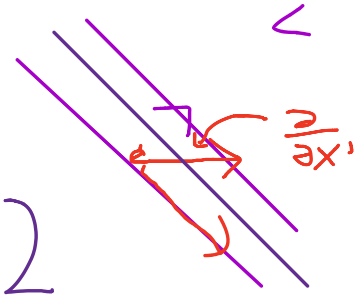
G-P convention: signed vol. of triangle.

The rest of the alg all works out fine; no inconsistencies

Cf G-P. Exer. 4.2: *10, 3.

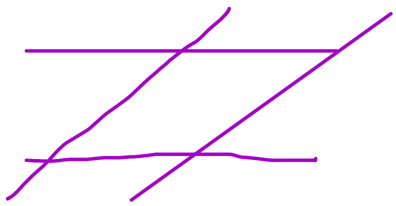
$\beta = 2\alpha$ $\alpha = (dx^1 \wedge dx^2)$

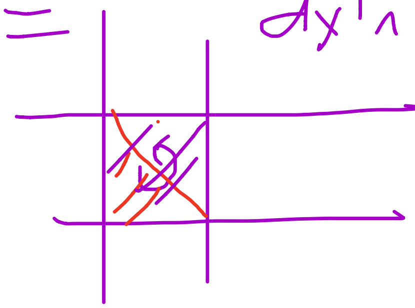
Burke.



$\beta(\partial_1) = 2$ $\qquad$ $\alpha\left(\frac{\partial}{\partial x^1}\right) = 1$

$dx^1 \wedge (dx^1 + dx^2) = dx^1 \wedge dx^2$





$$(dx^1 + dx^2)\frac{\partial}{\partial x^1} = \underbrace{dx^1(\partial_1)}_{1} + \underbrace{dx^2(\partial_1)}_{0}$$

see Gravitation, MTW for pics.

$\mathbb{V}$, $\mathbb{V}^* \otimes \mathbb{V}^* \leftarrow$ bilin. fns on $\mathbb{V}$

$$(\alpha_1 \otimes \alpha_2)(v_1, v_2) = \alpha_1(v_1)\,\alpha_2(v_2)$$

$$\mathbb{V} \underset{\mathbb{R}}{\otimes} \mathbb{W}, \quad \mathbb{V} \otimes \mathbb{V}^* \simeq \mathrm{Hom}(\mathbb{V})$$

$$\mathbb{V} \otimes \mathbb{V}^* \ni e \otimes \theta$$

$$\begin{aligned} e \otimes \theta : \mathbb{V} &\longrightarrow \mathbb{V} \\ v &\longmapsto \theta(v)e \end{aligned}$$

$$\begin{aligned} \mathbb{V}^* \otimes \mathbb{V}^* &= \text{Bilin maps on } \mathbb{V} \\ &= S^2(\mathbb{V}^*) \oplus \Lambda^2 \mathbb{V}^* \end{aligned}$$

We think of
$\Lambda^k \mathbb{V}^*$ as completely alternating k-linear fns on $\mathbb{V}$.

so:

$\omega \in \Lambda^k \mathbb{V}^*$; $\omega : \underbrace{\mathbb{V} \times \cdots \times \mathbb{V}}_{k\text{-times}} \longrightarrow \mathbb{R}$

$dx^i \wedge dx^j = -dx^j \wedge dx^i$

Case of $\mathbb{V} = \mathbb{R}^n$; use basis $dx^1, \ldots, dx^n$ for $\mathbb{V}^*$.

so basis for
$\Lambda^k \mathbb{V}^*$ is $dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

$I = i_1 < i_2 < \cdots < i_k$

any multi-index.
$\binom{n}{k}$ such.

$\sum_k \binom{n}{k} = 2^n$; $\dim \Lambda \mathbb{V}^* = 2^{\dim \mathbb{V}}$.

Lin alg,

$$\mathbb{V} \xrightarrow{L} \mathbb{W} \Rightarrow \mathbb{W}^* \xrightarrow{L^*} \mathbb{V}^*$$

$$\Rightarrow L^*: \Lambda^k \mathbb{W}^* \to \Lambda^k \mathbb{V}^*.$$

by $\omega \in$, $v_i \in \mathbb{V}$

$$(L^*\omega)(v_1, \ldots, v_k) = \omega(\underbrace{Lv_1}, \ldots, \underbrace{Lv_k}_{\mathbb{W}})$$

In particular

if $L: \mathbb{V} \to \mathbb{V}$ $\quad n = \dim \mathbb{V}$

get

$$L^*: \Lambda^n \mathbb{V}^* \to \Lambda^n \mathbb{V}^* \simeq \mathbb{R}$$

Now $\Lambda^n \mathbb{V}^*$ is 1-dim.

any basis $\mu =$ "vol. form"

eg. $\mathbb{R}^n$: $dx^1 \wedge \ldots \wedge dx^n$.

exer. $\boxed{L^*\mu = (\det L)\mu}$

any basis for $\mathbb{V}^*$

eg. $dx^1, \ldots, dx^n$

or $\theta^1, \theta^2, \ldots, \theta^2$

form $\theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^n = \mu$

Amethyst!

$\lambda \neq 0$

$$\mathbb{C}P^1 = (\mathbb{C}^2 \setminus 0) / \mathbb{C}^*$$

$\mathbb{C}$ λ's group action.

$$(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$$

look for $\mathbb{C}^*$-invariant fns

$$\frac{2 z_1 \bar{z}_2}{|z_1|^2 + |z_2|^2}, \quad \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}$$

$$X_2 + i X_3, \quad X_1$$

$$X_1^2 + X_2^2 + X_3^2 = 1$$

$$[z_1, z_2] \longrightarrow (x_1, x_2, x_3)$$

$$\mathbb{CP}^2 \simeq S^2$$

K-forms on $\mathbb{V}$:
Smooth maps $\mathbb{V} \xrightarrow[\omega]{} \Lambda^K \mathbb{V}^*$.

case $\mathbb{V} = \mathbb{R}^n$. $|I| = K$

e.g. $P(x,y)dx + Q(x,y)dy$

$$\omega = \sum \omega_I(x) dx^I$$

$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_K}$.
$x \in \mathbb{R}^n$.
$\omega_I(x)$ smooth.

~~define~~ ~~Ω~~ to set of all such:
$\Omega^K(\mathbb{R}^n)$.

$$\Omega^K(\mathbb{R}^n) \xrightarrow{d} \Omega^{K+1}(\mathbb{R}^n)$$

$= \sum \frac{\partial f}{\partial x^i} dx^i$

by: $d(f) = df, \quad f \in \Omega^0(\mathbb{R}^n)$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta.$$

How to figure out sign?

well: $ddx^i = 0$.

Exer $ddf = 0$ Do!

here next time

Only sensible def then.

$$d\left(\sum \omega_I \, dx^I\right) = \sum d\omega_I \wedge dx^I$$

This works.

Now you can work out signs.

Exer $d^2 = 0$ in complete generality.

$\Omega^k(M)$