Conventions on $\wedge$, $\otimes$ etc.

Is $(dx^1 \wedge dx^2)(\partial_1, \partial_2) = 1$, or $\frac{1}{2}$?

Guillemin - Pollack: $\frac{1}{2}$.
Abraham-Marsden, Morita, Bourbaki, Lee, ...: 1
Flanders: non-committal.

We will stick w/ "1". Then if $\alpha_1, \ldots, \alpha_K \in V^*$
$v_1, \ldots, v_K \in V$

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_K)(v_1, \ldots, v_K) = \det(\alpha_i(v_j))$$

eg. $K = 2$.

$$(\alpha_1 \wedge \alpha_2)(v_1, v_2) = \alpha_1(v_1)\alpha_2(v_2) - \alpha_1(v_2)\alpha_2(v_1)$$

$\Rightarrow$ rep. signed vol. of parallelipiped

G-P convention: signed vol. of triangle

The rest of the alg all works out fine; no inconsistencies

Cf G-P. Exer. 4.2: *10, 3.

We think of
$\Lambda^k V^*$ as completely alternating
k-linear fns on V.

so:

$\omega \in \Lambda^k V^*$; $\omega : \underbrace{V \times \cdots \times V}_{k\text{-linear}} \longrightarrow V$.

Case of $V = \mathbb{R}^n$; use basis $dx^1, \ldots, dx^n$
for V^*.

so basis for
$\Lambda^k V^*$ is $dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

$I = i_1 < i_2 < \ldots < i_k$

any multi-index.
$\binom{n}{k}$ such.

$\sum_k \binom{n}{k} = 2^n$; $\dim \Lambda V^* = 2^{\dim V}$.

Lin alg,

$$\mathbb{V} \xrightarrow{L} \mathbb{W} \Rightarrow \mathbb{W}^* \to \mathbb{V}^*$$

$$\Rightarrow L^* : \Lambda^k \mathbb{W}^* \to \Lambda^k \mathbb{V}^*.$$

by $(L^*\omega)(w_1, \dots, w_k) = \omega(Lw_1, \dots, Lw_k)$

In particular

if $L : \mathbb{V} \to \mathbb{V}$

get $L^* : \Lambda^n \mathbb{V}^* \to \Lambda^n \mathbb{V}^*$

Now $\Lambda^n \mathbb{V}^*$ is 1-dim.

any basis μ = "vol. form"

eg. $\mathbb{R}^n$: $dx^1 \wedge \dots \wedge dx^n$.

eg er. $\boxed{L^* \mu = (\det L)\mu}$

k-forms on V:
Smooth maps $V \xrightarrow[\omega]{} \Lambda^k V^*$.

case $V = \mathbb{R}^n$.

$$\omega = \sum \omega_I(x)\, dx^I$$

$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.
$x \in \mathbb{R}^n$.
$\omega_I(x)$ smooth.

~~define~~ Set of all such:
$\Omega^k(\mathbb{R}^n)$.

define d

$$\Omega^k(\mathbb{R}^n) \xrightarrow{d} \Omega^{k+1}(\mathbb{R}^n)$$

by: $d(f) = df$, $f \in \Omega^0(\mathbb{R}^n)$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta.$$

How to figure out sign?

well: $ddx^i = 0$.

Exer $ddf = 0$ Do!

Only sensible def then.

$$d\left(\sum \omega_I \, dx^I\right) = \sum d\omega_I \wedge dx^I$$

This works.
Now you can work out signs.

<u>Exer</u> $d^2 = 0$ in complete generality.