Feb 2, 2021 Part B 1 B

Sketch pf, Degree Formula.

Key: "Stack of records theorem"

Use Euler's identity for homogeneous polynomials

$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = \mathrm{m}\cdot p$$

to prove that 0 is the only critical value of p.]

***7.** (*Stack of Records Theorem.*) Suppose that y is a regular value of $f\colon X \longrightarrow Y$, where X is compact and has the same dimension as Y. Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$. Prove there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_i of x_i that are mapped diffeomorphically. Show that $f(X - \cup W_i)$ is compact and does not contain y.] See Figure 1-13.

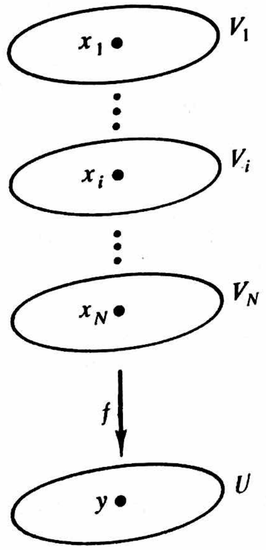

Figure 1-13

8. Let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a polynomial with complex coefficients, and consider the associated map $z \longrightarrow p(z)$ of the complex plane $\mathbf{C} \longrightarrow \mathbf{C}$. Prove that this is a submersion except at finitely many points. [HINT: Show that ... is compact. ...]

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At the heart of the degree formula lies the following theorem, which should remind you strongly of a fundamental property of degree.

Theorem. If $X = \partial W$ and $f: X \longrightarrow Y$ extends smoothly to all of W, then $\int_X f^*\omega = 0$ for every k-form ω on Y. (Here X and W are compact, all three manifolds are oriented and $k = \dim X = \dim Y$.)

Proof. Let $F: W \longrightarrow Y$ be an extension of f. Since $F = f$ on X,

$$\int_X f^*\omega = \int_{\partial W} F^*\omega = \int_W F^*\, d\omega.$$

But ω is a k-form on a k-dimensional manifold, so $d\omega = 0$. (All $k + 1$ forms on k-dimensional manifolds are automatically 0.) Q.E.D.

k-dimensional manifolds, then for every k-form ω on Y

$$\int_X f_0^*\omega = \int_X f_1^*\omega.$$

Proof. Let $F: I \times X \longrightarrow Y$ be a homotopy. Now

$$\partial(I \times X) = X_1 - X_0,$$

so

$$0 = \int_{\partial(I\times X)} (\partial F)^*\omega = \int_{X_1} (\partial F)^*\omega - \int_{X_0} (\partial F)^*\omega$$

(0 according to the theorem). But when we identify X_0 and X_1 with X, ∂F becomes f_0 on X_0 and f_1 on X_1. Q.E.D.

A local version of the degree formula around regular values is very easily established, and its proof shows most concretely the reason why the factor deg (f) appears.

Lemma. Let y be a regular value of the map $f: X \longrightarrow Y$ between oriented k-dimensional manifolds. Then there exists a neighborhood U of y such that the degree formula

$$\int_S f^*\omega = \deg(f)\int_Y \omega$$

is valid for every k-form ω with support in U.

Proof. Because f is a local diffeomorphism at each point in the preimage $f^{-1}(y)$, y has a neighborhood U such that $f^{-1}(U)$ consists of disjoint open sets $V_1, \ldots, V_N$, and $f: V_i \longrightarrow U$ is a diffeomorphism for each $i = 1, \ldots, N$ (Exercise 7, Chapter 1, Section 4). If ω has support in U, then $f^*\omega$ has support in $f^{-1}(U)$; thus

$$\int_X f^*\omega = \sum_{i=1}^{N} \int_{V_i} f^*\omega.$$

But since $f: V_i \longrightarrow U$ is a diffeomorphism, we know that

$$\int_{V_i} f^*\omega = \sigma_i \int_U \omega,$$

the sign σ_i being ± 1, depending on whether $f: V_i \longrightarrow U$ preserves or reverses orientation. Now, by definition, deg $(f) = \sum \sigma_i$, so we are done. Q.E.D.

Finally, we prove the degree formula in general. Choose a regular value y for $f\colon X \longrightarrow Y$ and a neighborhood U of y as in the lemma. By the Isotopy Lemma of Chapter 3, Section 6, for every point $z \in Y$ we can find a diffeomorphism $h\colon Y \longrightarrow Y$ that is isotopic to the identity and that carries y to z. Thus the collection of all open sets $h(U)$, where $h\colon Y \longrightarrow Y$ is a diffeomorphism isotopic to the identity, covers Y. By compactness, we can find finitely many maps $h_1, \ldots, h_n$ such that $Y = h_1(U) \cup \cdots \cup h_n(U)$. Using a partition of unity, we can write any form ω as a sum of forms, each having support in one of the sets $h_i(U)$; therefore, since both sides of the degree formula

$$\int_X f^*\omega = \deg(f) \int_Y \omega$$

are linear in ω, it suffices to prove the formula for forms supported in some $h(U)$.

So assume that ω is a form supported in $h(U)$. Since $h \sim$ identity, then $h \circ f \sim f$. Thus the corollary above implies

$$\int_X f^*\omega = \int_X (h \circ f)^*\omega = \int_X f^*h^*\omega.$$

As $h^*\omega$ is supported in U, the lemma implies

$$\int_X f^*(h^*\omega) = \deg(f) \int_Y h^*\omega.$$

Finally, the diffeomorphism h is orientation preserving; for $h \sim$ identity implies $\deg(h) = +1$. Thus the change of variables property gives

$$\int_Y h^*\omega = \int_Y \omega,$$

and

$$\int_X f^*\omega = \deg(f) \int_Y \omega,$$

as claimed.