Today Feb 2 : degree of a map
⇝ 4.8 GP also , 1.4 exer 7
"Hopf degree" ⟶ & 3.6, GP esp. 3.6 : Isotopy lemma.

Thurs Feb 4: Fröbenius w/ forms
(involutivity ⇔ integrability ⇔ differential ideal.

Lee , Ch. 19.
seen before?

$$X \xrightarrow{f} Y$$

X, Y cpt, $\partial X = \partial Y = \emptyset$, X, Y oriented.
$\mu = \mu_Y$ any n-form on Y
$y_0 \in Y$ any reg value.

Prop
"Degree Formula"

$$\int_X f^*\mu = \deg(f) \int_Y \mu \qquad (1)$$

where
$\deg(f) \in \mathbb{Z}$ &.

$$(2) \qquad \deg(f) = \sum_{\{x : f(x) = y_0\}} \operatorname{sgn} \det(df_x)$$

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On the integration part:

Recall $H^n(X) \simeq H^n(Y) \simeq \mathbb{R}$.

$$\begin{tikzcd} H^n(Y) \arrow[r, "f^*"] \arrow[d, "\wr"'] & H^n(X) \arrow[d, "\wr", "\int_X"'] \\ \mathbb{R} \arrow[r] & \mathbb{R} \end{tikzcd}$$

$\mu \mapsto \int_Y \mu$

a linear map.
must be mult by
some number.
$c(f)$, a homotopy
invariant:

$f_0 \sim f_1$ via $F: X \times I \to \mathbb{R}$

$\Rightarrow c(f_0) = c(f_1)$.

Surprise : $c(f) \in \mathbb{Z}$. !

here is where formula (2)
comes in.

$$T_x X \xrightarrow[\sim]{df_x} T_{y_0} Y. \quad ; f(x) = y_0$$

linear iso if y_0 reg
so, either preserves ~~serves~~ or reverses
the orientations of these vector spaces

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oriented

in coord's near x, y_0.

f given by $y^i = y^i(x^1, \dots, x^n)$

$$\det df = \det \frac{\partial y^i}{\partial x^j}$$

depends on coord choice
but sgn det df does not

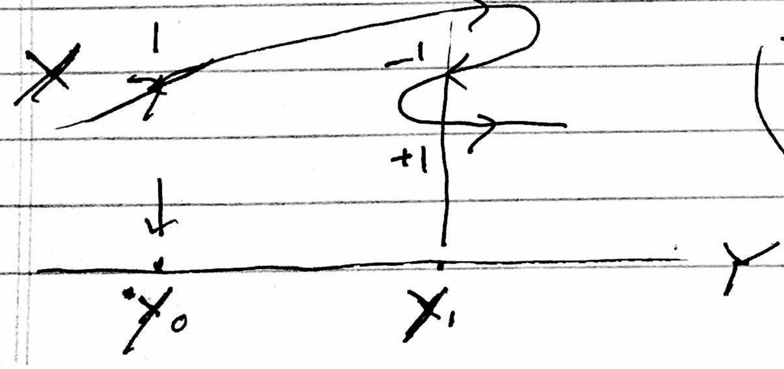
since we know what "positively oriented" means on X, Y.

Thus
$\operatorname{sgn} \det df_x = +1$, df_x orient pres.

-1 df_x orient reversing.

& formula (2) makes sense.

Surprise: indep. of choice of regular value.



typical pic see GP 3.6 or Milnor

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Eg of idea, but not of degree formula:

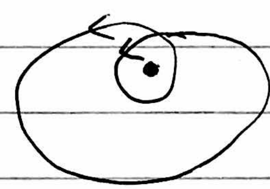
$\cancel{p(z) = z^m + a}$

1) $\gamma : S^1 \longrightarrow \mathbb{C} \setminus 0.$?

then $f = \frac{\gamma}{|\gamma|} : S^1 \longrightarrow S^1$

$\deg\left(\frac{\gamma}{|\gamma|}\right) =$ winding # of γ about 0 $\equiv w(\gamma)$

also works for



-2 ; $\int \gamma^* d\theta = w(\gamma)$

Similarly for $f : X^{n-1} \longrightarrow \mathbb{R}^n.$

if f misses p_0

& X^{n-1} is oriented cpt.

Then $\mathbb{R}^n$ has "solid angle form"

$dx^1 \wedge \cdots \wedge dx^n = dr \wedge r^{n-1} d\omega.$

$d\omega = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \lrcorner\, dx^1 \wedge \cdots \wedge dx^n.$

$n=2$: $\frac{x\,dy - y\,dx}{x^2+y^2} =$ "$d\theta$"

$\int_{S^n(r)} d\omega = \cancel{c(n)} = C(n)$

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So. $\deg\left(\frac{f}{|f|}\right) = \int_X f^* \omega$

if normalize: $\int_{S^{n-1}(\varepsilon)} \omega = 1.$

Other applications:

Pf of Fund Thm Alg.

see Milnor or GP.

presentation!

&

Argument principle.

$p(z)$ polyn. or mero. w/ isolated zeros & poles.

$p^* d\theta = \frac{1}{2\pi i} \operatorname{Im} \frac{dp}{p} = \frac{1}{2\pi i} \frac{p'(z)\,dz}{p(z)}$

& $\int_{\partial\Omega} p^* d\theta = \#$ zeros inside Ω,

$= \#$ zeros $- \#$ deg 1 poles

Ω