L17D Geodesics for Embedded p. 1

3) say $\Sigma \subset \mathbb{R}^N$ embedded submfd

w induced metric:

$$\langle v, w \rangle_s = v \cdot w,$$

$$v, w \in T_s \Sigma \subset \mathbb{R}^n.$$

eg: $S^2 \subset \mathbb{R}^3$.

Then $\gamma : I \to \Sigma$ is a const speed. geodesic

$$\Leftrightarrow \ddot{\gamma}(t) \perp T_\gamma \Sigma$$ at all points $\gamma(t)$ of γ.

In case $\dim \Sigma = N-1$ of an orientable hypersurface, eg: S^2.

have $\Sigma = \{ f = 0 \}$, unit normal field $= \vec{n} = \nabla f |_\Sigma$.

& eqn reads:

$$\begin{cases} \ddot{\gamma}(t) = f(t)\, n(\gamma(t)) \\ f(\gamma(t)) = 0 \end{cases}$$

L17D: Geod for Emb

Again $\left[\begin{array}{l}\Sigma \subset \mathbb{R}^N. \\ \text{geod. eqn.} \Leftrightarrow \ddot{\gamma} \perp T_\gamma \Sigma.\end{array}\right.$ ⟵ Theorem

pf simple case where Σ is a hypersurface of form $\Sigma = f^{-1}(0)$

eg $S^2 = \{x^2+y^2+z^2-1=0\}$

then $T_\gamma \Sigma = \operatorname{Ker} df(\gamma)$

use Lag. multipliers:

Min: $\int \frac{1}{2}|\dot{\gamma}|^2 + \lambda(t) f(\gamma(t))\, dt.$

cts Lag mult. to enforce ptwise constant $f(\gamma(t)) = 0$.

δ: $\int \langle \dot{\gamma}, \delta\dot{\gamma} \rangle + \lambda(t)\, df(\gamma)(\delta\gamma(t)).$

$\int$ by parts:

$$\int -\langle \ddot{\gamma}, \delta\gamma \rangle + \langle \lambda(t) \nabla f(\gamma(t)), \delta(\gamma) \rangle\, dt.$$

true $\forall$ variations $\Leftrightarrow$ $-\ddot{\gamma} + \lambda(t) \nabla f(\gamma(t)) = 0$

& $f(\gamma(t)) = 0$ ie γ a curve in Σ.

S^2 case:

take any ~~two u~~ o. n. frame u_1, u_2, u_3 for $\mathbb{R}^3$.

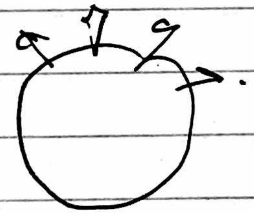
$\gamma(t) = \cos t\, u_1 + \sin t\, u_2$.

$\dot\gamma = -s\, u_1 + c\, u_2$

$\ddot\gamma = -c\, u_1 - s\, u_2$.

But $f(p) = |p|^2 - 1$

$\nabla f(p) = 2p$.



$\ddot\gamma = -\gamma \leftarrow$ neg unit norm.

$= -\frac{1}{2}\nabla f(p)$

✓

L17 D Geod Embedde 4

$\langle X, Y \rangle = tr(XY^T) = tr(YX^T)$
$= \sum X_{ij} Y_{ij}$

thus $M_n \simeq \mathbb{R}^{n^2}$ o.n. coord x_{ij}.

$O(n) \curvearrowright M_n$
$X \mapsto AY$

$tr\langle AX, AY\rangle = tr\, AXY^TA^T$
$= tr(XY^TA^TA)$
$= tr(XY^T)$

also
$tr(XA, YA) = tr(X, Y) \quad A \in O(n)$

$\Rightarrow I \mapsto O(n)\cdot I$
yields an isometric immersion.
get: bi-inv. metric on $O(n)$

<u>Exer</u>: e^{tA} ~~or~~ $A^T = -A$

are the geod thru I.

<u>Exer</u> diam $(O(n)) = \pi\sqrt{n}$.

$\sqrt{n}$ from $tr\, I = \|I\|^2 = n$.