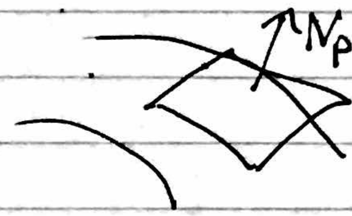
L17.M 1

Case of induced metric

$\Sigma^2 \hookrightarrow \mathbb{R}^3$;

$ds^2 = \langle\ ,\ \rangle\big|_{T\Sigma}$.



Find o.n. frame $e_1, e_2, N = e_3$ defined near p for computations.

will see

1) $\nabla_X Y = (D_X Y)^T$

where $T: \mathbb{R}^3 \longrightarrow T_p\Sigma$ for $p \in \Sigma$

is the orthogonal projection:

$$Z^T = Z - \langle Z, N\rangle N.$$

2) $ds^2 = (\theta^1)^2 + (\theta^2)^2$; θ^i dual to e_i

3) $\underbrace{\det(dN : T_p\Sigma \to T_p\Sigma) = K(p)}_{\text{"theorem egregium"}}$

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Write $\vec{x} : \Sigma \hookrightarrow \mathbb{R}^3$ for embedding

choose coord u, v so.

$$u, v \longmapsto \vec{x}(u,v) \in \mathbb{R}^3.$$

then

$$d\vec{x} = \frac{\partial \vec{x}}{\partial u} du + \frac{\partial \vec{x}}{\partial v} dv$$

$$= e_1 \otimes \theta^1 + e_2 \otimes \theta^2.$$

[really: $Id_p : T_p\Sigma \to T_p\Sigma$ expressed in terms of two different dual basis. $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$; du, dv & e_1, e_2, θ^1, θ^2.]

Then

$$\|d\vec{x}\|^2 = \langle d\vec{x}, d\vec{x} \rangle$$

$$= (\theta^1)^2 + (\theta^2)^2.$$

$$= g_{uu} du^2 + g_{uv} du dv + g_{vv} dv^2$$

$$g_{uu} = \left\langle \frac{\partial \vec{x}}{\partial u}, \frac{\partial \vec{x}}{\partial u} \right\rangle \text{ etc.}$$

$$= ds^2 \quad \checkmark.$$

Showing (2).

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set $de_i = \omega_i^j e_j$

where d is usual 'd' of a vector valued function, so "d" = "D".

Claim: $\omega_i^i = 0$, $\omega_i^j = \omega_j^i$

eg: $\langle e_1, e_1 \rangle = 1$

so $d\langle e_1, e_1 \rangle = 0$

$$= \langle de_1, e_1 \rangle + \langle e_1, de_1 \rangle$$

$$= \langle \omega_1^j e_j, e_1 \rangle + \langle e_1, \omega_1^j e_j \rangle$$

$$= 2\omega_1^1$$

So for example.

$$de_1 = \omega_1^2 e_2 + \omega_1^3 e_3.$$

Thus require..

$$\nabla_X e_1 = \omega_1^2(X) e_1$$

$$= (de_1(X))^T$$

since $e_3 = N$.

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Exercise: Verify that

$$X, Y \mapsto (D_X Y)^T$$

for $X, Y \in \mathfrak{X}(\Sigma)$ "$\subset$" $\mathfrak{X}(\mathbb{R}^3)$

verifies the axioms of the Levi-Civita connection.

This will establish (1).

LIFM #5.

$$d\vec{x} = \frac{\partial \vec{x}}{\partial u} du + \frac{\partial \vec{x}}{\partial v} dv$$

$$= \theta^1 e_1 + \theta^2 e_2 .$$

view as a vector valued one form.

θ^i o.n. coframe

e_i o.n frame.

with $e_3 = N =$ unit normal to surface.

$$d^2\vec{x} = \frac{\partial^2 \vec{x}}{\partial v \partial u} dv \wedge du + \frac{\partial^2 \vec{x}}{\partial u \partial v} du \wedge dv = 0$$

$$= d\theta^1 e_1 + \theta^1 de_1 + d\theta^2 e_2 + \theta^2 de_2 .$$

$$= 0 .$$

But : $de_1 = \omega_1^2 e_2 + \omega_1^3 e_3$

$de_2 = \omega_2^1 e_1 + \omega_2^3 e_3 .$

No e_i in de_i since $d\langle e_i, e_i \rangle = 1$
$= 2\langle e_i, de_i \rangle .$

$$\cancel{\Rightarrow 0 = \omega \wedge \theta^2 e_1 + \theta^1(\omega_1^2 e_2 + \omega_1^3 e_3)}$$

$$\cancel{- \omega \theta^1 e_2 + \theta^2(\omega_2^1 e_1 + \omega_2^3 e_3)}$$

$$\cancel{= (\omega \wedge \theta^2 + \theta^2 \wedge \omega_2^1) e_1 + (-\omega \wedge \theta^1 -}$$

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Expanding out $d^2\vec{x} = 0$, get:

$d\theta^1 e_1 + \theta^1(\omega_1^2 e_2 + \omega_1^3 e_3) + d\theta^2 e_2 + \theta^2(\omega_2^1 e_1 + \omega_2^3 e_3)$

$\wedge$

$0 = (d\theta^1 + \theta^2 \wedge \omega_2^1)e_1 + (d\theta^2 + \theta^1\omega_1^2)e_2 + (\theta^1\wedge\omega_1^3 + \theta^2\wedge\omega_2^3)e_3$

so

$d\theta^1 + \theta^2 \wedge \omega_2^1 = 0$

$d\theta^2 + \theta^1\wedge\omega_1^2 = 0$

$\cancel{d\theta}$

$\cancel{d}\ \theta^1\wedge\omega_1^3 + \theta^2\wedge\omega_2^3 = 0$

really?: $\underbrace{\cancel{d\theta^3 + \theta^1\wedge\omega_1^3 + \theta^2\wedge\omega_2^3 = 0}}$

$\cancel{d\theta^3 = 0}$ defines $T\Sigma$

$\cancel{d\theta^3} \equiv 0$

or $d\theta^1 = -\theta^2\wedge\omega_2^1 = +\omega\wedge\theta^2$, if $\omega = \omega_2^1$

$d\theta^2 = -\theta^1\wedge\omega_1^2 = +\omega_1^2\wedge\theta^1$, $= -\omega_1^2$

since $d\langle e_1, e_2\rangle = 0$.

What about ∇?

$\nabla_X Y = (D_X Y)^{\cancel{\ }T}$

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$$de_1 = \omega_1^2 e_2 + \omega_1^3 e_3$$
$$= -\omega e_2 + \omega_1^3 e_3.$$

$$dde_1 = 0$$
$$= -d\omega\, e_2 - \omega\, de_2 + d\omega_1^3 e_3 + \omega_1^3 de_3.$$

each term, e_1, e_2, e_3 must be zero. Focus on e_2.

$$\langle e_2, de_2 \rangle = 0,$$

$$de_3 = \omega_3^1 e_1 + \omega_3^2 e_2.$$

e_2 term:

$$-d\omega + \omega_1^3 \omega_3^2 = 0.$$

$$d\omega = +\omega_1^3 \omega_3^2$$

$$= -\omega_3^1 \wedge \omega_3^2$$

well up to sign!

$$d\omega = \omega_3^1 \wedge \omega_3^2$$ hmm.

sign does not matter!

$\det(-A) = \det A$ in 2 dim.

$$A = dN_p = \omega_3^1 e_1 + \omega_3^2 e_2$$
$$-A = -\omega_3^1 e_1 - \omega_3^2 e_2$$