L17 1

A. <u>Algorithm</u> for computing connection one-form; curvature, for a <u>Riemannian surface</u>.

B. <u>Meaning</u> of connection, curvature.

A. <u>Algorithm</u>. 1) Given ds^2 on Σ^2 find an o.n. coframe θ^1, θ^2 s.t. $ds^2 = (\theta^1)^2 + (\theta^2)^2$. This can be done by Graham-Schmidt or completing the square.

2) Solve:

$$d\theta^1 = +\omega \wedge \theta^2$$
$$d\theta^2 = -\omega \wedge \theta^1$$

i.e. $d\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix} = \begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix} \wedge \begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix}$.

ω is the connection one form.

L17 2.

"B"

if e_1, e_2 are the dual o.n. frame then

$$\nabla \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ +\omega & 0 \end{pmatrix} \begin{matrix} e_1 \\ e_2 . \end{matrix}$$

define the <u>Levi-Civita</u> connection ∇. ; i.e. $\nabla_X e_1 = -\omega(X) e_2$ etc.

3) Compute

$$d\omega = K\theta^1 \wedge \theta^2 .$$

Then $K =$ Gaussian curvature.

Example: Model <u>case</u> $S^2(1)$

has $K = 1$

L17 2' Eg S^2.

Case of S^2.

$\vec{x}(\varphi,\theta) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$

$$d\vec{x} = \frac{\partial\vec{x}}{\partial\varphi}d\varphi + \frac{\partial\vec{x}}{\partial\theta}d\theta$$

compute: $\left|\frac{\partial\vec{x}}{\partial\varphi}\right|^2 = 1$, $\left|\frac{\partial\vec{x}}{\partial\theta}\right|^2 = \sin^2\varphi$

$$\left\langle \frac{\partial\vec{x}}{\partial\varphi}, \frac{\partial\vec{x}}{\partial\theta}\right\rangle = 0$$

$$\Rightarrow \langle d\vec{x}, d\vec{x}\rangle = d\varphi^2 + \sin^2\varphi\, d\theta^2$$
$$= ds^2.$$

so: $(\theta^1) = d\varphi$; $\theta^2 = \sin\varphi\, d\theta$.

$d\theta^1 = 0$

$d\theta^2 = -\cos\varphi\, d\varphi \wedge d\theta$.

But $d\theta^1 = \omega\wedge\theta^2 \Rightarrow \omega = f\,d\theta$.

$d\theta^2 = -\omega\wedge\theta^1 = -f\,d\theta\wedge d\varphi = +f\,d\varphi\wedge d\theta$

$\Rightarrow f = -\cos\varphi$

$\Rightarrow \omega = -\cos\varphi\, d\theta$

$d\omega = +\sin\varphi\, d\varphi\wedge d\theta$; $\theta^1\wedge\theta^2 = d\varphi\wedge\sin\varphi\, d\theta$
$= \sin\varphi\, d\varphi\wedge d\theta$

$\Rightarrow K = 1$.

L17 3', Eg const curv. L17 3;

Example

if $ds^2 = dr^2 + f(r)^2 d\theta^2$.

$\theta^1 = dr$

$\theta^2 = f(r) d\theta$.

$d\theta^1 = 0 = \omega \wedge \theta^2 \Rightarrow \omega = g\,d\theta$

$d\theta^2 = f' dr \wedge d\theta \Rightarrow -\omega \wedge \theta^1$

$\Rightarrow -g\, d\theta\, dr = f' dr\, d\theta$.

$\Rightarrow \cancel{\omega}\ g = -f'$

$\Rightarrow \omega = -f' d\theta$

$\Rightarrow d\omega = -f'' dr \wedge d\theta$.

But $\theta^1 \wedge \theta^2 = f\, dr \wedge d\theta$.

$$\Rightarrow \boxed{K = -\frac{f''}{f}} \quad \checkmark$$

Model examples: $f = r$: Euclidean

$f = \sin r$ Spherical $K = +1$

$f = \sinh r$ hyperbolic $K = -1$.