Why manifolds with mechanics?

or

Why 'mechanics on manifolds'?

A basic example: ~~Ge~~ (1) Motion of a "free rigid body" as a geodesic flow on the Lie group $G = SO(3)$.

(2) less basic: ~~How a cat~~ The cat's problem as a problem in gauge theory

(3) some quantum mechanics.

General principles

Q a "configuration space" - a manifold,

"mechanics" happens on either on TQ or T^*Q

the two pictures of mechanics are called "Lagrangian" & "Hamiltonian"

they are related by the "Legendre transformation:

$$\begin{tikzcd} TQ \arrow[rr, "Leg"] \arrow[dr] & & T^*Q \arrow[dl] \\ & Q & \end{tikzcd}$$

Simplest cases.

$Q = \mathbb{R}^3$.

$q(t)$ curve.

ODE:

$$m\ddot{q} = F(q, \dot{q}) \qquad \text{"} F = ma \text{"}$$

Newton's eqns.

so $(q, \dot{q}) \in T\mathbb{R}^3 = T^*\mathbb{R}^3$.

Example "F"

$F = -\nabla V(q)$; $\quad V : \mathbb{R}^3 \to \mathbb{R}$.

so

$\nabla V : \mathbb{R} \to \mathbb{R}^3$.

so

$$(N) \longrightarrow \boxed{m\ddot{q} = -\nabla V(q)} \; (N)$$

Conserved energy:

$$E = \frac{1}{2} m |\dot{q}|^2 + V(q)$$

<u>Prop</u> IF $q(t)$ sat $m\ddot{q} = -\nabla V(q)$

then $E(q(t), \frac{dq}{dt}) = \text{const}$.

<u>Pf</u> $\frac{d}{dt} E = m \langle \dot{q}, \ddot{q} \rangle + \langle \dot{q}, \nabla V(q) \rangle$

$$= \langle \dot{q}, \underbrace{m\ddot{q} + \nabla V(q)}_{0} \rangle.$$

0. !

Lagrangian:

$$L(q,\dot q) = \frac{1}{2} m |\dot q|^2 - V(q).$$

<u>Action</u>: $A(q(\cdot)) = \int_I L(q(s),\dot q(s)) d.$

where $q: I \longrightarrow \mathbb{R}^3$. , $I = [a,b]$

<u>Prop</u>: [Principle of least action].

if V is smooth ~~& q(t) sa~~

then $q(t)$ sat. (N) ~~iff~~

<u>iff</u>

$$dA|_{\Omega_{\{q(a),q(b)\}}}(q(\cdot)) = 0 \qquad \text{"}\delta A = 0\text{"}$$

where

$\Omega_{\{q(a),q(b)\}}$ = absolutely cts paths $\sigma: I \to \mathbb{R}^3$ s.t. $\sigma(a) = q(a)$, $\sigma(b) = q(b)$.

Various questions arise.

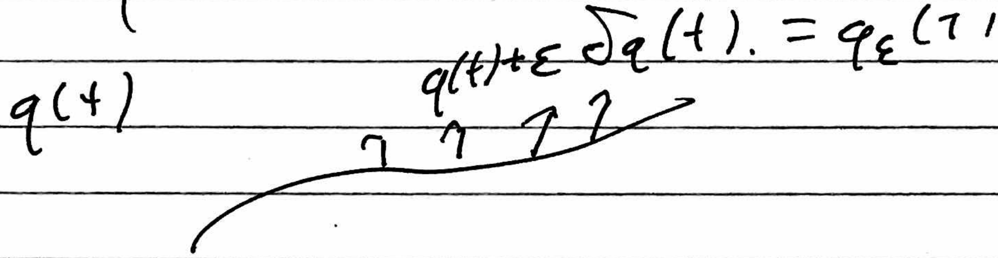
a) how do I differentiate in path space?

. what topology to use on path space?

etc etc.

See: Gelfand - Fomin, "Calculus of Variations" for example.

Here is the "seat of the pants" pf of "$\Longleftarrow$"



$$A(q_\varepsilon(\cdot)) = \int \tfrac{1}{2} m\, |\dot q_\varepsilon(t)|^2 - V(q_\varepsilon(t))\, dt$$

so

$$\frac{d}{d\varepsilon} A(q_\varepsilon(\cdot)) = \int m \langle \dot q_\varepsilon(t), \delta \dot q_\varepsilon(t)\rangle - \langle \nabla V(q_\varepsilon), \delta q(t)\rangle$$

Use:

$$\frac{d}{dt}\langle \dot q, \delta q\rangle = \langle \ddot q, \delta q\rangle + \langle \dot q, \delta \dot q\rangle$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A(q_\varepsilon(\cdot))$$

$$\cancel{= \int -m\langle \ddot{q}(t), \delta q(t)\rangle -}$$

so:

$$\langle \dot{q}, \delta\dot{q}\rangle = -\langle \ddot{q}, \delta q\rangle + \frac{d}{dt}\langle \dot{q}, \delta q\rangle$$

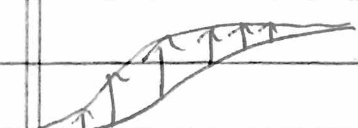
$\Rightarrow$

$$\left.\frac{d}{d\varepsilon}A(q_\varepsilon)\right|_{\varepsilon=0} = \int_a^b -m\langle \ddot{q}, \delta q\rangle + m\langle \dot{q}, \delta q\rangle\Big|_a^b$$

$$- \int_a^b \langle \nabla V(q), \delta q\rangle$$

$$= \int_a^b \langle -m\ddot{q}(t) - \nabla V(q(t)), \delta q(t)\rangle$$

$$+ m\langle \dot{q}(b), \underset{0}{\cancel{\delta q(b)}}\rangle - m\langle \dot{q}(a), \underset{0}{\cancel{\delta q(a)}}\rangle.$$



since $q_\varepsilon(a) \equiv q(a)$, $q_\varepsilon(b) = q(b)$