L13 1 Feb 16 2021

$\mathfrak{g} = \text{Lie}(G) = T_e G$

to get Lie bracket on $\mathfrak{g}$
identify $\mathfrak{g}$ w left-invariant v-flds on G.

$\mathfrak{g} \cong \mathfrak{X}^L(G)$

matrix case: $X \longrightarrow X^L(g) = gX$.

general case.
$X \in T_e G$ define $X^L(g) = (dL_g)_e X$

so $X^L : G \longrightarrow TG$.

Exer. 1) $L_h^* X^L = X^L$.

2) X^L is the unique left invariant vector field s.t. $X^L(e) = X$.

Lie bracket:

$$[X, Y] = [X^L, Y^L](e)$$

N.B. $L_h^*[X^L, Y^L] = [L_h^* X^L, L_h^* Y^L]$

$$= [X^L, Y^L]$$

so $\mathfrak{X}^L(G)$ is closed under Lie bracket

Matrix case:

$$[X, Y] = XY - YX.$$

Pf $X^L(g) = gX$.

Now X^L, Y^L are restrictions of vector fields on $\mathbb{M}_N$ to $G \subset GL(N) \subset \mathbb{M}_N$.

Formula for bracket of restricted vector fields:

S submfld of $\mathbb{V}$; $i: S \hookrightarrow \mathbb{V}$

$X, Y : \mathbb{V} \to \mathbb{V}$

$X, Y \parallel S$ ie: $X(s), Y(s) \in T_s S$
$\forall s \in S$

Then $[i^* X, i^* Y] = i^*[X, Y]$

& $[X, Y] = [X^i \partial_i, Y^j \partial_j]$

$= (X^i \partial_i Y^j - (Y \leftrightarrow X)) \partial_j$

$= DY \cdot X - DX \cdot Y$

on $\mathbb{V}$

Us:
& $Y^L(g) = gY$.

linear in g.

$\therefore DY \cdot X = \frac{d}{d\varepsilon}(g(I+\varepsilon X))Y$

$= gXY$.

&
$[X^L, Y^L](g) = g(XY - YX)$

so $[X^L, Y^L](e) = XY - YX$.

$\exp: \mathfrak{g} \to G$.

Let Φ_t^X be flow of X^L

then $\Phi_t^X \circ L_h = L_h \circ \Phi_t^X$

set $\exp(X) = \Phi_1^X(e)$.

$= g(1)$ where $\dot{g} = gX$
$g(0) = e$.

Matrix case:

$$\exp(X) = \text{matrix exponential}$$
$$= e^X$$
$$\doteq 1 + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots$$

General case:

$t \mapsto \exp(tX)$ is a 1-parameter subgroup in G.
ψ_t.

$$\frac{d}{dt}\Big|_{t=0} \exp(tX) = X$$

ie: $\psi : \mathbb{R} \longrightarrow G$ a homo.

& $\psi(t+s) = \psi(t)\psi(s)$

$\psi(0) = e$.

& all 1-p subgroups are of this form.