L12 B

Lie algebra:

$$\mathfrak{g} = T_e G \hookrightarrow \mathfrak{X}(G)$$

by taking each $\xi \in \mathfrak{g}$ & *left-extending*:

$$\xi^L(g) = g\xi \quad \text{, matrix case}$$

$$= (dL_g)_e\,\xi \quad \text{general.}$$

1) $\xi^L(e) = \xi$ $\qquad G \xrightarrow{L_h} G$

2) $L_h^* \xi^L = \xi^L$

pf: $(L_h^* \xi^L)(g) = dL_h^{-1}\,\xi(L_h g)$

$$= (dL_h)^{-1}_{hg}\,\xi(hg)$$

$$\begin{cases} = h^{-1} h g \xi & \text{matrix case} \\ = g\xi & \\ = dL_h^{-1} (dL_{hg})_e \xi & \text{gen} \\ = \cancel{dL_h^{-1}}\, \cancel{dL_h}\, dL_g \xi & \end{cases}$$

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So $\mathfrak{g} \simeq \mathfrak{X}^{left}(G)$

Lie bracket: as per ~~left~~ $\mathfrak{X}$

N.B: $L_g^* \xi = \xi \quad L_g^* \zeta = \zeta$

$\Rightarrow L_g^* [\xi, \zeta] = [L_g^* \xi, L_g^* \zeta]$

$= [\xi, \zeta]$

so bracket takes left inv.
to left inv.

Matrix case:
$X \in \mathfrak{g} \subset M_N$.

$X^L(g) = gX$

Compute $[X^L, Y^L] = (XY - YX)^L$.

Pf vector fields on a vector space
so $X : V \longrightarrow V$.
then: claim $[X,Y](v) = (D_X Y - D_Y X)(v)$

since, eg $X = X^i \partial_i, \quad Y = Y^j \partial_j$

$[X,Y] = [X^i \partial_i, Y^j \partial_j] = X^i \partial_i Y^j \partial_j - X \leftrightarrow Y$

which is $DY \cdot X - DX \cdot Y$

$$= D_X Y - D_Y X.$$

given $X^L(g) = gX, \quad g \in G \subseteq M_N$

Then $D_v Y = vY, \quad , v \in M_N.$

so

$$(D_X Y)_g = X(g) Y$$
$$= gXY.$$

similarly $(D_Y X)_g = gYX$

& $[X^L, Y^L](g) = g(XY - YX)$

$$= [X, Y]^L(g)$$

where $[X, Y] = XY - YX$

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properties of $[X, Y] = -[Y, X]$

$[X, [Y, Z]] + \circlearrowleft$

& bilinearity.

Struc constants:

choose basis E_i of $\mathfrak{g}$.

$$[E_i, E_j] = \sum_k c_{ij}^k E_k$$

θ^i dual basis for $\mathfrak{g}^*$.

Then $(\theta^i)^L$ coframe for G.

$$(\theta^i)^L \in \Gamma(T^*G)$$

write θ^i for $(\theta^i)^L$

Claim:

$$X^{-1}dX = \sum (\theta^i)^L E_i(e)$$

$$d(\theta^i) = -\sum c^i_{jk}\, \theta^j \wedge \theta^k.$$

Pf.

$$d\theta^i(E_j, E_k) = E_j\, \theta^i(E_k) - E_k\, \theta^i(E_j)$$

$$-\theta^i([E_j, E_k])$$

$$= 0 - 0$$

$$-\theta^i\left(\sum c_{jk}^l E_l\right)$$