L 10 B. = LIIA, p1 last time

vector bundles.

subbundles

$$\begin{array}{ccc} T^*Q^n & & TQ^n \\ \cup & & \cup \\ \mathcal{I} = D^\perp & \xleftrightarrow{\text{pairing.}} & D \end{array}$$

locally: $\theta^1, \theta^2, \ldots, \theta^c$ $X_1, X_2, \ldots, X_K$

$K + c = n.$

$\theta^i(X_j) = 0.$

integrable: <u>dual version, Fröb,</u> $d\mathcal{I} \subset \mathcal{I}$. $\xLeftrightarrow{\text{Th'm}}$ Fröb. $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.

$\mathcal{I}$ = ideal in $\Omega(Q)$ generated by θ

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eg: a single 1-form

$d\theta \in \mathcal{I} \iff d\theta = \theta \wedge \beta, \ \beta \in \Omega^1$

$\iff \theta \wedge d\theta = 0.$

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Notation $\mathcal{D} = \Gamma(D)$ - v-fields
$\mathcal{D}^\perp = \Gamma(D^\perp)$ - one forms.
$\mathcal{I}$ = ideal gen. by $\mathcal{D}^\perp$ in Ω

Key fact:
if $X, Y \in \mathcal{D}$, $\theta \in \mathcal{D}^\perp$
then

$$d\theta(X,Y) = -\theta([X,Y])$$

<u>Pf</u>: $d\theta(X,Y) = X[\underbrace{\theta(Y)}_{0}] - Y[\underbrace{\theta(X)}_{0}]$
$- \theta([X,Y])$

<u>Thus</u>: $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D} \Leftrightarrow d\theta_q$ annihilates D_q at each $q \in Q$.

Now use some exterior linear algebra to show:
$\Leftrightarrow d\theta \in \mathcal{I}$.

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<u>Exterior</u> linear algebra.

$\mathbb{V}$, $\mathbb{V}^*$.

$\mathcal{I} \subset \Lambda^* \mathbb{V}^*$

an ideal generated by covectors $\theta^1, .., \theta^c$

set $S^\perp = \text{Span}\{\theta^1, .., \theta^c\} \subset \mathbb{V}^*$

$S = \{s \in \mathbb{V} : \theta^i(s) = 0\} \subset \mathbb{V}$.

<u>~~Claim~~ So</u>: $\mathcal{I} = \langle S^\perp \rangle$

<u>Claim</u>: $\mathcal{I} = \{\alpha \in \Lambda^* \mathbb{V}^* : i_S^* \alpha = 0\}$

where $i_S : S \to \mathbb{V}$, so $i_S^* : \Lambda^* V^* \to \Lambda^* S^*$.

Sketch pf: 1) $i_S^*(\theta^i \wedge \beta) = i_S^* \theta^i \wedge i_S^* \beta$
$= 0$.

2) Complete $\theta^1, .. \theta^c$ to a basis $\theta^1, .., \theta^c, \eta^1, .. \eta^r$, $r + c = n$
$= \dim \mathbb{V}$

for $\mathbb{V}^*$. Use that
$i_S^* \eta^i$ generate $\Lambda^* S^*$.

Have proved

D involutive $\Leftrightarrow [\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$
$\Leftrightarrow d\mathcal{I}_{\mathcal{D}} \subset \mathcal{I}_{\mathcal{D}}$
$\Leftrightarrow D$ integrable:

Normal form: if D is invol., rank r then through each pt $\exists$ coords $x^1, \ldots x^r, x^{r+1}, \ldots x^n$

so 1) $D = \text{Span}\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r}\}$.

2) $\mathcal{I}_D = \langle dx^i, i = r+1, \ldots n\}$

3) the leaves of D are given by $x^i = \text{const}_i \quad i = r+1, \ldots, n$.

Pf by induction on r.
$r = 1$. Straightening lemma

Induction: See EDS, linked now in web page, Bryant et al p. 24. "Frobenius thm"

Case of a single Pfaffian eqn.

$\mathcal{D}^\perp = \langle \theta \rangle \qquad \theta(q) \neq 0.$

$d\mathcal{I}_D \subset \mathcal{I}_D \iff \theta \wedge d\theta = 0$

$\iff \exists\ f, g$ functions > 0.

$\theta = f\,dg.$

the leaves of $\{\theta = 0\}$

$g = \text{const}.$

On to Cheyenne.

Break.

Who presents next?