

L 10

$B. = \mathbb{C} \parallel A, p1$

last time

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vector bundles.

subbundles

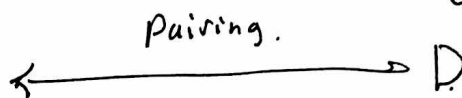
$T \mathbb{C}^n$

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$U$

$U$

$\mathcal{I} = \mathcal{D}^\perp$



$\mathcal{D}$

locally:

$\theta^1, \theta^2, \dots, \theta^c$

$X_1, X_2, \dots, X_k$

$k+c=n.$

$\theta^i(X_j) = 0.$

integrable:

Dual version, Fröb.  
 $d\mathcal{I} \subset \mathcal{I}.$



Fröb.  
 $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}.$

$\mathcal{I} = \text{ideal in } \Omega(\mathbb{C})$

generated by  $\theta$

eg: a single 1-form

$d\theta \in \mathcal{I}$

$\Leftrightarrow d\theta = \sum \theta \wedge \beta, \beta \in \Omega^1$

$\Leftrightarrow \theta \wedge d\theta = 0.$

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Notation

$\mathcal{D} = \Gamma(\mathcal{D})$  - v-fields

$\mathcal{D}^\perp = \Gamma(\mathcal{D}^\perp)$  - one forms

$\mathcal{I} =$  ideal gen. by  $\mathcal{D}^\perp$   
in  $\mathcal{D}$

Key fact:

if  $X, Y \in \mathcal{D}$ ,  $\theta \in \mathcal{D}^\perp$

then

$$d\theta(X, Y) = -\theta([X, Y])$$

$$\begin{aligned} \text{pf: } d\theta(X, Y) &= X[\theta(Y)] - Y[\theta(X)] \\ &\quad - \theta([X, Y]) \end{aligned}$$

Thus:  $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D} \Leftrightarrow d\theta_q$  annihilates  $\mathcal{D}_q$  at each  $q \in Q$ .

Now use some exterior linear algebra  
to show:

$$\Leftrightarrow d\theta \in \mathcal{I}$$

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Exterior linear algebra.

$V, V^*$ .

$$\mathcal{I} \subset \bigwedge^r V^*$$

an ideal generated  
by covectors  $\theta^1, \dots, \theta^c$

set

$$S^\perp = \text{Span}\{\theta^1, \dots, \theta^c\} \subset V^*$$

$$S = \{s \in V : \theta^i(s) = 0\} \subset V$$

So

~~Claim:~~  $\mathcal{I} = \langle S^\perp \rangle$

Claim:  $\mathcal{I} = \{\alpha \in \bigwedge^r V^* : i_S^r \alpha = 0\}$

where  $i_S: S \rightarrow V$ , so  $i_S^r: \bigwedge^r V^* \rightarrow \bigwedge^r S^*$ .

Sketch pf: 1)  $i_S^r(\theta^i \wedge \beta) = i_S^r \theta^i \wedge i_S^r \beta = 0$

2) Complete  $\theta^1, \dots, \theta^c$  to a basis  
 $\theta^1, \dots, \theta^c, n^1, \dots, n^r$ ,  $r+c = n = \dim V$

for  $V^*$ . Use that  $i_S^r n^i$  generate  $\bigwedge^r S^*$ .

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Have proved

$$D \text{ involutive} \Leftrightarrow [D, D] \subset D$$
$$\Leftrightarrow d\mathcal{I}_D \subset \mathcal{I}_D$$

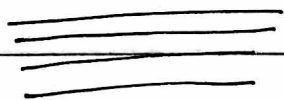
$\Leftrightarrow D$  integrable:

Normal form if  $D$  is invol., rank  $r$   
then through each pt  $\exists$  coord  
 $x^1, \dots, x^r, x^{r+1}, \dots, x^n$

so 1)  $D = \text{span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r} \right\}$

2)  $\mathcal{I}_D = \langle dx^i, i = r+1, \dots, n \rangle$

3) the leaves of  $D$  are given  
by  $x^i = \text{const}; i = r+1, \dots, n.$



Pf by induction on  $r$ .

$r=1$ . Straightening lemma

Induction: See EDS, linked  
now on web page, Bryant et al.  
p. 24. "Frobenius theorem"

Case of a single Pfaffian  
eqn.

$$\mathcal{D}^\perp = \langle \Theta \rangle \quad \Theta(q) \neq 0.$$

$$d\mathcal{I}_D \subset \mathcal{I}_D \Leftrightarrow \Theta \wedge d\Theta = 0$$

$$\Leftrightarrow \exists f, g \text{ functions} \\ > 0.$$

$$\Theta = f dg.$$

the leaves of  $\{\Theta = 0\}$

$$g = \text{const.}$$


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on to Cheyenne.

~~\*~~

Break.

Who presents next?