θ a one-form.

(*) $\underline{\theta = 0}$ "a Pfaffian eqn"

so far: we've seen forms as

1) things you $\int$ over
2) " " use to compute cohomology

(*) is a 3rd use

What's a "solution" to a Pfaffian eqn?

let Σ^k be a k-dim submfd

an soln is an immersion or an embedding.

$i : \Sigma^k \hookrightarrow M.$

$i^*\theta = 0.$

Picture $\{\theta = 0\}$ defines a "field of hyperplanes"

$D_m \subset T_m M$

$D_m = \ker \theta_m = \{v \in T_m M : \theta_m(v) = 0\}$

L10

Pic here for

$\theta := dz - y\,dx = 0$

so: a field of two-planes on $\mathbb{R}^3$

what is the max. dim "K" of a

K-dim'l sol'n $i : \Sigma^K \hookrightarrow \mathbb{R}^3$.

$K \overset{?}{=} 0, 1, 2, \cancel{3}$

$\theta \neq 0$.

⋮

Prop $K < 2$.

Pf if $\exists\ \Sigma^2 \hookrightarrow \mathbb{R}^3$ $i^*\theta = 0$.

then $d i^*\theta = i^* d\theta = 0$

But $d\theta = dx \wedge dy \neq 0$....

Okay but why must $i^*d\theta \neq 0$

Use graph coord:

basis for $D = \ker \theta$

is $\frac{\partial}{\partial x} + y \frac{\partial}{\partial z}$

$\frac{\partial}{\partial y}$.

Let (u,v) be coord on this fictitious Σ.

$$u, v \xmapsto{\ i\ } i(u,v) = x(u,v), y(u,v), z(u,v)$$

$$\downarrow$$

$$x(u,v), y(u,v)$$

~~But~~ Must have $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$.

since the linear map $\pi(x,y,z) = (x,y)$ $\quad d\pi \quad \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}$

$$\downarrow$$

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

is a linear surjection. $\det(d(i \circ \pi))$

<u>Eg ctd</u>

I N.B. $\Big[\underbrace{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}}_{X_1}, \underbrace{\frac{\partial}{\partial y}}_{X_2}\Big] = -\frac{\partial}{\partial z}$.

$X_1, X_2, [X_1, X_2]$
everywhere lin. indep.

$\Rightarrow \{X_1, X_2\}$ Not invol.

$\Rightarrow D$ not integrabl.

Integrable: thru every pt p there passes a smooth embedded submfd Σ w/ $T_q\Sigma = D_q$.
$= \mathrm{Span}\, X_1(q), X_2(q)$.

Single Pfaffian eqn $\Longleftrightarrow$ codim 1 DCTQ

$\theta \wedge d\theta = 0 \iff D$ integrable.