Quadratic forms, bilinear forms and symmetric matrices.

Throughout $\mathbb{V}$ denotes a real vector space and $\mathbb{V}^*$ its dual. We set $dim(\mathbb{V}) = n$. Understanding the case $dim(\mathbb{V}) = n2$ well usually equips you for the general case.

In the back of your mind $\mathbb{V}$ is to be thought of as the tangent space to a manifold at a point. The important thing about $\mathbb{V}$ is that although it has a basis, it has no prefered basis: no prefered "xy" coordindates for $n = 2$. (Think of the plane $x + y + z = 0$. Why choose one basis over the other?) Geometrically then, when $n = 2$, $\mathbb{V}$ is a plane with a marked origin, but no preferred axes.

1. A single quadratic form.

Definition: A quadratic form on $\mathbb{V}$ is a homogeneous quadratic map $Q : \mathbb{V} \to \mathbb{R}$. This means

$$Q(\lambda\vec{v}) = \lambda^2 Q(\vec{v}),$$

and that, relative to any basis, Q is a quadratic polynomial in the coordinates.

Planar case, $n = 2$ Choose a basis e_1, e_2 for $\mathbb{V}$ then

$$Q(xe_1 + ye_2) = ax^2 + 2bxy + cy^2$$

for some numbers a, b, c. Thus, the three numbers a, b, c determine the quadratic form. Since

$$Q(xe_1 + ye_2) = (x \quad y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

we call

$$Q_{\mathcal{B}} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

the matrix of Q relative to the basis $\mathcal{B} = \{e_1, e_2\}$, or with respect to the coordinates x, y for $\mathbb{V}$.

The tensor notation for Q is

$$Q = adx^2 + 2bdxdy + cdy^2.$$

where $dx, dy \in \mathbb{V}^*$ are the dual basis to e_1, e_2 and we revert to manifold notation and write $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}$. The products $dx^2, dxdy, dy^2$ are emphatically NOT exterior ('$\wedge$) products of one-forms, but rather symmetric products of basis one-forms.

Symmetric product of one- forms. We want to understand the cross term $dxdy$ occuring above it helps to expand out $(dx + dy)^2$ and stare at the cross term. It also helps to think of quadratic forms as their equivalents: bilinear symmetric forms. To turn a bilinear symmetric form $\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ into a quadratic form is a simple thing:

$$Q_\beta(v) = \beta(v, v).$$

The reverse of this procedure is called "polarization" and we will get to it shortly. For right now, accept that the space of bilinear symmetric forms on $\mathbb{V}$ is precisely the same space as the space of all quadratic forms on $\mathbb{V}$.

On the one hand, if $\alpha \in \mathbb{V}^*$ is any one-form, then $\alpha^2 \in S^2(\mathbb{V}^*)$ must be the quadratic form sends $v, w \in \mathbb{V}$ to $\alpha(v)\alpha(w)$. So we understand the meaning of $(du + dv)^2$. But formally $(du + dv)^2 = du^2 + 2dudv + dv^2$ and we understand the meaning of du^2 and dv^2. Thus $dudv = \frac{1}{2}((du+dv)^2 - du^2 - dv^2)$. This requires that if $\theta, \alpha \in \mathbb{V}^*$ then their 'symmetric product" is the bilinear form

$$\alpha \odot_s \beta : v, w \mapsto \frac{1}{2}(\alpha(v)\beta(w) + \alpha(w)\beta(v).$$

Exercise 1. Verify that $a = Q(e_1), c = Q(e_2)$.

Challenge: How do you recover the coefficient b from applying Q to vectors $v, w \in \mathbb{V}$?

What does $Q_{\mathcal{B}}$ look like in a different basis, or, how does Q change under change of coordinates?

Example 1. We can convert $Q(x,y) = xy$ to $Q(u,v) = u^2 - v^2$ by making the invertible linear change of coordinates

$$x = (u+v), y = (u-v)$$

Let $\mathcal{B}' = \{f_1, f_2\}$ be another basis for $\mathbb{V}$ and let $Q_{\mathcal{B}'} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ be the matrix of Q relative to this new matrix.

Exercise 2 Prove

$$Q_{\mathcal{B}'} = BQ_{\mathcal{B}}B^t \qquad (*)$$

where B is the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$. Thus:

$$f_1 = B_1^1 e_1 + B_1^2 e_2$$

$$f_2 = B_2^1 e_1 + B_2^2 e_2;$$

$$B = \begin{pmatrix} B_1^1 & B_1^2 \\ B_2^1 & B_2^2 \end{pmatrix}$$

Returning to example 1, the matrix for xy is $Q_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ while the matrix for $u^2 - v^2$ is $Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Verify that $BQ_1B^t = Q^t$ with $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Theorem 1.1 (Basic theorem of real quadratic forms, 2 dimensional case.)**.** *If $\mathbb{V}$ is a two-dimensional vector space and Q is a quadratic form on $\mathbb{V}$ then $\mathbb{V}$ is equivalent to exactly one of: $x^2 + y^2, -(x^2+y^2), x^2 - y^2, x^2, -x^2$ or 0. In other words, there are linear coordinates x, y on $\mathbb{V}$ such that $Q = \epsilon_1 x^2 + \epsilon_2 y^2, \epsilon_i \in \{-1, 0, 1\}$.*

In matrix terms, this theorem asserts that there is a basis $\mathcal{B}$ for $\mathbb{V}$ such that $Q_{\mathcal{B}}$ is exactly one of

$$Q_{\mathcal{B}} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In tensor notation, at a point, this theorem asserts that

$$Q = \pm(dx^2 + dy^2), dx^2 - dy^2, \pm dx^2, 0.$$

Definition 1.2. The rank of Q is the rank of $Q_{\mathcal{B}}$. The index of Q is a listing of the number of +1's and -1's in the normal form (*), so it is $(2,0), (0,2), (1,1), (1,0), (0,1), (0,0)$ respectively for $x^2 + y^2, -x^2 - y^2, x^2 - y^2, x^2, -x^2, 0$.

The basic theorem thus asserts that two quadratic forms on $\mathbb{V}$ are equivalent (can be changed one into the other by a linear invertible change of coordinates) if and only if their ranks and indices are the same.

It is worth meditating on the contrast between this basic theorem of quadratic forms, and the basic theory of linear operators $L : \mathbb{V} \to \mathbb{V}$. Most linear operators have eigenvalues λ_1, λ_2. The eigenvalues are continuous invariants: real numbers, with any (unordered) pair of real numbers occuring. Operators with different eigenvalues are *not equivalent*. When the eigenvalues are distinct, they form a complete set of invariants: two diagonalizable operators are equivalent if and only if their eigenvalues are equal.

On the other hand, the basic theorem of quadratic forms asserts that a quadratic form has nothing like eigenvalues. The only invariant of a quadratic form is its rank and index and these invariants form a finite set.

At the heart of this difference between linear transformations and quadratic forms is the difference in transformation laws. The matrix $L_{\mathcal{B}}$ of a linear transformation L transforms under change of basis according to

$$L_{\mathcal{B}'} = B L_{\mathcal{B}} B^{-1}$$

which is to be contrasted with (*) above:

$$Q_{\mathcal{B}'} = B Q_{\mathcal{B}} B^t \qquad (*).$$

Example 2. Hessian.

Let $f : \mathbb{V} \to \mathbb{R}$ be a smooth function. Then its 2nd order Taylor expansion reads:

$$f(q) = f(0) + df_0(q) + \frac{1}{2} d^2 f_0(q) + o(|q|^2).$$

The second order term, $\frac{1}{2} d^2 f_0(q)$ is the Hessian of f at 0 and is a quadratic form.

2. Bilinear pairing of a quadratic form

Pretend that $Q(v) = \langle v, v \rangle$ for some inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{V}$. A bit of algebra shows that we can recover the inner product from Q:

Exercise 3. [Polarization] If $Q(v)\langle v, v \rangle$, then expand out $Q(V + W), Q(V - W)$ to show that

$$\langle V, W \rangle = \frac{1}{4}[Q(V + W) - Q(V - W)]$$

Definition 2.1. A symmetric bilinear form on $\mathbb{V}$ is a map $\beta : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ which is symmetric: $\beta(V, W) = \beta(W, V)$ for all $V, W \in \mathbb{V}$ and bilinear, meaning linear in each slot $\beta(cV + Z, W) = c\beta(V, W) + \beta(Z, W)$ and similarly for $\beta(W, cV + Z)$.

A symmetric bilinear form β gives rise to a quadratic form

$$Q(V) = \beta(V, V).$$

On the other hand, the polarization exercise 3 applies to any quadratic form Q to yield a symmetric bilinear form β defined by

$$\beta(V, W) = \frac{1}{4}[Q(V + W) - Q(V - W)].$$

These processes are inverses: the quadratic form of the bilinear form β of Q is again Q. Quadratic forms and symmetric bilinear forms are realy the same thing!

Exercise 4. The bilinear form associated to the quadratic form $Q(x, y) = xy$ is $\beta((x_1, y_1), (x_2, y_2) = \frac{1}{2}(x_1 y_2 + x_2 y_1)$

Exercise 5. Return to notation from the beginning of these notes: $\mathcal{B} = \{e_1, e_2\}$ a basis for $\mathbb{V}$, $Q_{\mathcal{B}} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ the matrix of Q relative to this basis. Show that the bilinear form β associated to Q is given by

$$\beta(x_1 e_1 + y_1 e_2, x_2 e_1 + y_2 e_2) = (x_1 \quad y_1) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

3. Positive definite inner products.

In case $Q > 0$ whenever $v \neq 0$, we say that the associated bilinear form β is an inner product. And we say that Q is positive-definite. In this case Q equivalent to $x^2 + y^2$, or, written in tensor form, $dx^2 + dy^2$. It is a simple algebraic exercise to verify that Q is positive definite if and only if its matrix $Q_{\mathcal{B}}$ as above satisfies $ac - b^2 > 0$ (notice closeness to the discriminant) and $a, c > 0$.

Exercise 6. Let β be the bilinear form associated to Q, and suppose that β is an inner product. Let $Q_{\mathcal{B}}$ the matrix of Q relative to the basis $\{\mathcal{B}\} = \{e_1, e_2\}$ for $\mathbb{V}$. Prove that $Q_{\mathcal{B}} = Id$ if and only if $\{e_1, e_2\}$ are a β- orthonormal basis.

4. Pairs of quadratic forms

Now suppose we are given two quadratic forms Q_1, Q_2 on the same vector space $\mathbb{V}$. All of a sudden we have invariants! In geometry, the two quadratic forms are the 1st and 2nd fundamental forms.

Theorem 4.1. *Let Q_1, Q_2 be two quadratic forms on $\mathbb{V}$ with Q_1 positive definite. Let β_1, β_2 be the corresponding bilinear forms. Then there exists a unique linear operator $S : \mathbb{V} \to \mathbb{V}$ such that $\beta_2(v, w) = \beta_1(v, Sw)$ holds for all $v, w \in \mathbb{V}$. Moreover, S is β_1-symmetric: $\beta_1(Sv, w) = \beta_1(v, Sw)$ for all $v, w \in \mathbb{V}$.*

Definition 4.2. We say that S "intertwines" Q_2 and Q_1 if it relates them as in the theorem.

Theorem 4.3. *Any (β_1) symmetric operator S is diagonalizable over the reals: there exists a (β_1)- orthonormal basis e_1, e_2 such that $Se_1 = k_1 e_1, Se_2 = k_2 e_2$, $k_i \in \mathbb{R}$*

Corollary 4.4. *Given a pair of quadratic forms, one of which is positive definite, there is a coordinate system which simultaneously diagonalizes them:*

$$Q_1 = x^2 + y^2$$
$$Q_2 = k_1 x^2 + k_2 y^2.$$

The eigenvalues k_1, k_2 are the continuous invariants of the pair Q_1, Q_2.

Exercise. Fix the inner product β_1 on $\mathbb{V}$. Show that the linear space of all quadratic forms on $\mathbb{V}$ is linearly isomorphic to the space of all β_1-symmetric linear transformations $S : \mathbb{V} \to \mathbb{V}$. What is the dimension of this space?

Our situation.

$Q_1 = ds^2|_p = I_p$ is the 1st fundamental form, which is just the restriction of the inner product of $\mathbb{R}^3$ to the tangent space $\mathbb{V} = T_p\Sigma$.

Q_2 is the second fundamental form.

S, the intertwining operator, is the shape operator, which is minus the differential of the Gauss map at p. $S = -dN_p$.

The eigenvalues k_1, k_2 are called the "principal curvatures". Euler showed, following the 'min-max interpretation" immediately below that these numbers (depending on p) are the curvatures of certain 'extremal planar-curve sections" of the surface Σ. See section 5.3 of text and immediately below.

5. Finding and interpreting the eigenvalues. Min-max.

Form the quotient

$$q = Q_2/Q_1 : \mathbb{V} \setminus 0 \to \mathbb{R}.$$

Note the quotient is homogeneous of degree 0: $q(\lambda v) = q(v)$, $\lambda \neq 0 \in \mathbb{R}, v \neq 0 \in \mathbb{V}$ and as such, is a function on the space of lines through the origin in $\mathbb{V}$. This space of lines forms a circle.

Theorem 5.1. *The critical values of the quotient $q = Q_2/Q_1$ are precisely the eigenvalues of S, i.e. k_1, k_2. The corresponding critical lines, being the span of the nonzero vectors v for which $dq_v = 0$ are the eigenlines: the spans of the (two) eigenvectors of S.*

Proof 1. Use the previous corollary and compute.

Proof 2. Use calculus and the computation. $dQ_i(p)(w) = 2\beta_i(p, w)$.

Proof 3. Argue that extremizing q is the same as extremizing Q_2 subject to the constraint that $Q_1 = 1$. Use Lagrange multipliers.

Because the circle is two dimensional, there will always be two critical points for any function on it, a max and a min. This yields

$$k_1 = min_v Q_2(v)/Q_1(v) = min_{\{v:Q_1(v)=1\}} Q_2(v).$$

and

$$k_1 = max_v Q_2(v)/Q_1(v) = max_{\{v:Q_1(v)=1\}} Q_2(v).$$

We have been writing our first fundamental form, or metric (at a point) as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

If e_1, e_1 are the basis associated to the coordinates u, v, so that du, dv are the dual basis to e_1, e_2 then the inner product associated to ds^2 is

$$\beta(u_1e_1 + v_1e_2, u_2e_1 + v_2e_2) = (u_1 \quad v_1)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}.$$

The second fundamental form is traditionally written

$$II = Ldu^2 + 2Mdudv + Ndv^2.$$

REcall our basic theorem: the surface near p can uniquely be represented as a graph over the tangent plane $T_p\Sigma$. For linear algebraic convenience, think of $T_p\Sigma$ as a plane through the origin, so that $p + T_p\Sigma$ is the geometric tangent plane. Then what this basic graph representation theorem asserts is that there is a function $f : T_p\Sigma \to \mathbb{R}$, unique up to choice of normal, such that the surface near p is the graph of f: which is to say, any point q on Σ sufficeintly close to p is uniquely expressible in the form:

$$q = (p + v) + f(v)\vec{N}(p)$$

where $\vec{N}(p)$ is the normal vector to Σ at p. We have:

$$II_p = Hess(f_p).$$