REFERENCES. Henri Cartan, *NOtions d'algébre différentielle; applications aux groupes de Lie et aux variétés où opère un groupe de Lie.* Colloque du Topologie, C.B.R.M., Bruxelles, 15-27, (1950). Reprinted as Appendix 2 in the book "Supersymmetry and equivariant de Rham theory" by Guillemin and Sternberg, which is the other reference for this note, particularly their section 2.2, titled The Language of Superalgebra.

The exterior algebra over a vector space and the module of differential forms on a smooth manifold are both examples of *graded commutative superalgebras.* The exterior differential, interior product and Lie derivative are all examples of superderivations on this superalgebra.

Definition 1. *A graded algebra A over a field is an algebra A over that field which is graded as a vector space: $A = \oplus_{k=0}^{\infty} A_k$ and whose multiplication satisfies $A_r A_s \subset A_{r+s}$, i.e. whenever $\alpha \in A_r, \beta \in A_s$ then $\alpha\beta \in A_{r+s}$ We write $deg(\alpha) = r$ if $\alpha \in A_r$.*

We will assume that A has an identity 1. Necessarily $1 \in A_0$ since $1\beta = \beta$.

Definition 2. *A supercommutative algebra is a graded algebra such that $\alpha\beta = (-1)^{rs}\beta\alpha$ where $r = deg(\alpha), s = deg(\beta)$. We will call an element even or odd depending on whether its degree is even or odd.*

Definition 3. *Let A be a graded commutative superalgebra. A derivation is a linear map $\delta : A \to A$ which respects the grading and satisfies $\delta(\alpha\beta) = \delta(\alpha)\beta \pm \alpha\delta(\beta)$ with the $\pm$ sign to be discussed momentarily. We say that δ has degree r if $\delta(A^p) \subset A^{p+r}$ for all p. Then the sign $\pm$ is taken to be $(-1)^{rdeg(\alpha)}$. A derivation is called even or odd depending on whether its degree is even or odd.*

Examples. The exterior algebra $A = \Lambda^*\mathbb{V}^*$ of a vector space $\mathbb{V}$ is a graded commutative superalgebra, with $A_r = \Lambda^r\mathbb{V}^*$ the space of degree r forms. The algebra of differential forms $\Omega^*(M)$ over a manifold M is a graded commutative superalgebra $A_r = \Omega^r(M)$.

The operator i_v of interior product on $\Lambda^*\mathbb{V}$ by a vector $v \in \mathbb{V}$ is an odd derivation of degree -1. The operator of interior product i_v by a smooth vector field v is an odd derivation of $\Omega(M)$ of degree -1. The exterior differential is an odd derivation on $\Omega(M)$ of degree $+1$

Pull-back by a linear isomorphism $F : \mathbb{V} \to \mathbb{V}$ defines an automorphism of $\Lambda^*\mathbb{V}$. Pull-back by a smooth diffeomorphism $F : M \to M$ defines an automorphism of $\Omega(M)$.

Lemma 1. *If $\psi_t, t \in \mathbb{R}$ is a one-parameter family of automorphisms of a superalgebra then its derivative $d\psi_t/dt|_{t=0}$ is a degree 0 derivation of A*

Example. The Lie derivative $L_X = d\phi_t^*/dt|_{t=0}$ is a degree 0 derivation arising this way, where ϕ_t is the flow of X.

Lemma 2. *The linear space of all derivations of a graded superalgebra form themselves a graded superalgebra, graded by degree, with multiplication being composition. In particular, the product of two odd derivations is even.*

Example. di_X and $i_X d$ are even degree 0 derivations.

Observation: If A_0 and A_1 generate A then the value of a derivation is determined by its values on A_0 and A_1

Theorem 1 (Cartan's magic formula)**.**

$$L_X = di_X + i_X d$$

Remark. Cartan's Magic formula can also be written $L_X = (d + i_X)^2$.

Proof. Verify on Ω^0 and Ω^1. On Ω^0: $L_X f = df(X) = i_X df = i_X df + d(i_X f)$ since $i_X f = 0$.

On Ω^1. Use that $\phi^* d = d\phi^*$ for any diffeomorphism ϕ. Applying this fact to $\phi = \phi_t$ and differentiating in t get that $L_X d = dL_X$.

Thus: $L_X(fdg) = (L_X f)dg + fdL_X g = X[f]dg + fd[X[g]]$. On the other hand $(i_X d + di_X)(fdg) = i_X d(fdg) + d(fi_X dg) = i_X df \wedge dg + d(fX[g]) = df(X)dg - dg(X)df + fdX[g] + X[g]df = X[f]dg - X[g]df + fd[X[g]] + X[g]dg = X[f]dg + fd[X[g]]$. Since any one-form is a sum of one forms of the form fdg the result now follows.

Fundamental Differential Geometry identies

$$d^2 = 0 \tag{1}$$

$$i_X^2 = 0 \tag{2}$$

$$di_X + i_X d = 0 \tag{3}$$

$$L_X L_Y - L_Y L_X = L_{[X,Y]} \tag{4}$$

$$L_X i_Y - i_Y L_X = i_{[X,Y]} \tag{5}$$

$$dL_X - L_X d = 0 \tag{6}$$

Moreover suppose that $\phi : M \to M$ a diffeomorphism. Then its pull-back $\phi^* : \Omega(M) \to \Omega(M)$ is an algebra isomorphism. We have

$$d\phi^* = \phi^* d \tag{7}$$

The inverse of ϕ^* is the push-forward $\phi_* : \Omega(M) \to \Omega(M)$. Now push-forward also acts on the space $\chi(M)$ of vector fields. Then

$$\phi_* i_Y \phi^* = i_{\phi_* Y} \tag{8}$$

$$\phi_* L_Y \phi^* = L_{\phi_* Y} \tag{9}$$

Eq (4) follows from eq (9) by setting $\phi = \phi_t = exp(tX)$ and differentiating with respect to t. (5) follows from (8) in the same way. Eq (4) says that $X \mapsto L_X$ is a representation of the Lie algebra $\chi(M) = \Gamma(TM)$ on the vector space $\Omega(M)$.

There is a succinct way to encode almost all of these identities.

Definition 4. *An endomorphism of the graded algebra A is a linear map $L : A \to A$ which respects the grading of A in the sense that there is an r, independent of k such that $L(A_k) \subset A_{k+r}$. In this case we say that L has degree r and write $deg(L) = r$. If L, M are two endomorphisms of A with $deg(L) = r$ and $deg(M) = s$ then we define the supercommutator of L and M to be*

$$[L, M] = LM - (-1)^{rs} ML.$$

Summarizing Identities (1)-(6) in super language Observe that d, i_X are odd derivations while L_X is even. Then, with the exception of eq (4) and (5) every one of the identities (1)-(6) is a statement that the two operators involved in that equation supercommute.

Example $[d, d] = dd - (-1)^{1*1} dd = 2dd$, so to say that d supercommutes with itself is to say that $d^2 = 0$.

Remark. In the more standard definition of "supercommutative algebra" the algebra is $\mathbb{Z}/2\mathbb{Z}$ graded rather than $\mathbb{Z}$-graded. To go from a $\mathbb{Z}$ graded algebra to a $\mathbb{Z}/2\mathbb{Z}$ graded one, use the standard mod 2 map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and split A into two parts, "even" and "odd" where the even part consists of the sum of the A_k for k even and the odd part consists of the sum of A_k for k odd.

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Let us combine all the i_X's into a linear space $\mathfrak{g}_{-1}$, the L_X's into a linear space $\mathfrak{g}_0$ and d to span a one-diimensional linear space. Then $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ forms a subalgebra of $Der(\Omega(M))$. Moreover this subalgebra is a Lie supersubalgebra.

Look up the definition of "Lie superalgebra". Verify that $Der(A)$ is a Lie superalgebra with respect to the supercommutator. Verify that $\mathfrak{g}$ is a Lie superesubalgebra of $Der(\Omega(M))$.