■ <u>4.4.12</u> α a one-form

s.t. $\oint_c \alpha = 0$ $\forall$ closed paths

c.

Show α is exact: $\alpha = df$

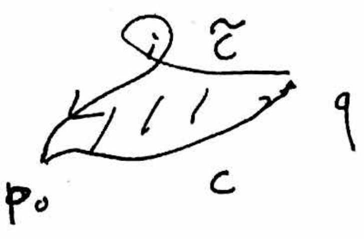
for some fn f.

1) Fix a pt. $p_0 \in M$. Assuming M is connected, choose a path c connecting q to p_0. Set

$$f(q) = \int_c \alpha.$$

This f is well defined since if $\tilde{c}$ is any other path from p_0 to q

then $\int_c \alpha - \int_{\tilde{c}} \alpha = \int_{\tilde{c}^{-1} * c} \alpha \stackrel{\Delta}{=} 0$



Δ $\tilde{c}^{-1} * c$ is a closed path since it returns & starts at p_0

2) $df = \alpha$.

Let $q \in M$, $v \in T_q M$.

$\boxed{\text{We are to show } df_q(v) = \alpha(q)(v).}$

Represent v by a path $c(t)$ thru q. Then

$$df(q)(v) = \frac{d}{dt}\Big|_{t=0} f(c(t))$$

~~Bu~~ Now, let $c_t = c([0,t])$, a path from q to $c(t)$. ~~Then~~.

~~f6~~ Choose <u>any</u> path γ_0 from p_0 to q.

~~Set~~ Then $c_t * \gamma_0$ is a path from p_0 to $c(t)$ ~~fa~~ so that

$$f(c(t)) = \int_{c_t * \gamma_0} \alpha$$

$$= \int_{c_t} \alpha + \int_{\gamma_0} \alpha$$

and $\int_{\gamma_0} \alpha = f(q) = f(c(0))$.

So

$$\left.\frac{d}{dt}\right|_{t=0} f(c(t)) = \lim_{t\to 0} \frac{1}{t}\left(f(c(t)) - f(c(0))\right)$$

$$= \lim_{t\to 0} \frac{1}{t} \int_{c_t} \alpha.$$

Now: $\int_{c_\varepsilon} \alpha = \int_0^\varepsilon \left(\alpha(c(s)) \cdot \frac{dc}{ds}\right) ds$

general def of integrating one forms on manifolds

So $\left.\frac{d}{dt}\right|_{t=0} f(c(t))$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \left(\left(\alpha(c(s)) \cdot \frac{dc}{ds}(s)\right)\right) ds.$$

Recall analysis: if g cts

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\varepsilon g(x)\,dx = g(0).$$

So: $$df(q)\,v = \overbrace{\alpha(c(0))}^{\text{"}g(0)\text{"}} \cdot v.$$
$$= \alpha(q) \cdot v.$$

QED