Plan:

2/12: Integration pairing. Define singular homology and thus cohomology by duality. The deRham isomorphism. A bit of Hodge theory, time permitting

2/19 and 24. Frobenius, the dual version. Chow, the dual version. A bit of EDS

2/26. Homogeneous space intro. G acts transitively on M $\implies$ M = G/H A few examples: spheres, projective space, Grassmannians, hyperbolic space. The space forms

3/3 and 3/5. Structure equations: for surfaces. For curves. For Riemannian manifolds. Perhaps for submanifolds. Cartan's lemma. The frame bundle.

3/10 and 3/12: beginning Riemannian geometry

3/17: how this course relates to my work.

2/12. A compact oriented k-dimensional sub-manifold $\Sigma \subset M$ defines a linear functional on k-forms ω by

$$\omega \mapsto \int_\Sigma \omega.$$

We would like this to induce a functional

$$H^k_{deR}(M) \to \mathbb{R}$$

So suppose that $\tilde{\omega}$ and ω represent the same cohomology class: $\tilde{\omega} - \omega = d\beta$ In order for this functional to be well-defined on cohomology we need that $\int_\Sigma \omega = \int_\Sigma \tilde{\omega}$ or

$$\int_\Sigma d\beta = 0$$

By Stokes' theorem $\int_\Sigma d\beta = \int_{\partial\Sigma} \beta$. So as Σ is without boundary: $\partial\Sigma = \emptyset$ this map is well defined modula exact $k-1$ forms and so descends to cohomololgy.

Why do we need $d\omega = 0$ for the map?

Definition 1. *Two oriented compact k-dimensional manifolds Σ and $\tilde{\Sigma}$ are cobordant in M if there is an oriented $k+1$-dimensional manifold-with-boundary N embedded in M such that $\partial N = \Sigma - \tilde{\Sigma}$*

Lemma 1. *The linear functionals $\omega \mapsto \int_\Sigma \omega$ and $\omega \mapsto \int_{\tilde{\Sigma}} \omega$ on the space of closed k-forms a are equal if Σ and $\tilde{\Sigma}$ are cobordant.*

Proof. Let N be the cobordism. Then $\int_\Sigma \omega - \int_{\tilde{\Sigma}} \omega = \int_{\partial N} \omega = \int_N d\omega = 0$, provided ω is closed.

This suggests defining the dual to cohomology as the space of cobordism classes of oriented sub manifolds. This definition is not the standard one of homology. (I am not sure exactly what it gives.) Instead : go to singular homology; smooth singular homology.... (Historically: we had simplicial homology then singular homology then smooth singular homology then singular cohomology then the dedham theorem.)

Simplicial homology. You look it up.

Singular homology: based on chains. Chains based on simplices in M. The simplices can be smooth or just continuous If they are smooth we can integrate forms over them.

Every compact smooth manifold can be triangulated. If the manifold is oriented then the boundary of the triangulation triangulates the boundary of the manifold....

Theorem 1 (deRham theorem)**.** *Integration over smooth singular k-chains defines an isomorphism from H^k_{deR} and $H_k(M, \mathbb{R})^*$ for M compact and oriented.*