1. Riemannian surfaces

Let Q be an oriented Riemannian surface. An *admissible coframe* $\theta = (\theta^1, \theta^2)$ is an oriented orthonormal basis for T^*Q. This means that $\theta^1 \wedge \theta^2$ is the area form (defined by the orientation and metric) and that the metric is $ds^2 = (\theta^1)^2 + (\theta^2)^2$. Equivalently, a basis for T_p^*Q is an oriented orthonormal coframe if and only if it is dual to some oriented orthonormal basis $\{e_1, e_2\}$ for T_pQ.

Any two oriented orthonormal coframes $\theta = (\theta^1, \theta^2)$ and $\tilde{\theta} = (\tilde{\theta}^1, \tilde{\theta}^2)$ are related by a rotation:

$$\begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}. \tag{1}$$

It follows that the set B of all oriented orthonormal coframes forms a circle bundle $\pi : B \to Q$, the circle being $G = SO(2)$, parameterized by ϕ. B is the G-structure, which encodes all the data of our oriented Riemannian surface. (See section **??** for the formal definition of a G-structure over a manifold.)

Given a local section $\theta : U \to B$ (that is, a smooth family of orthonormal coframes defined in a neighborhood U of Q), equation 1 expresses any other coframe $\tilde{\theta}$ defined in that neighborhood. Hence equation 1 defines a local trivialization $B_U \cong U \times G$ by sending (q, g) to $g^{-1}(\theta(q))$.

Any two-form on Q is of the form $f\theta^1 \wedge \theta^2$ for some function f. Thus $d\theta^1 = c_1\theta^1 \wedge \theta^2$, $d\theta^2 = c_2\theta^1 \wedge \theta^2$ for some functions c_1, c_2. Cartan tells us to rewrite this in the form

$$d\theta^1 = -\omega \wedge \theta^2, \quad d\theta^2 = +\omega \wedge \theta^1. \tag{2}$$

Viewed as a linear equation for the one-form ω, this equation has a unique solution. We ask the reader to check that the solution is $\omega = -c_1\theta^1 - c_2\theta^2$.

Suppose that $(\tilde{\theta}^1, \tilde{\theta}^2)$ is another coframe, related to (θ^1, θ^2) by the transformation 1. Differentiating this transformation we compute

$$d\tilde{\theta}^1 = -\tilde{\omega} \wedge \tilde{\theta}^2, \quad d\tilde{\theta}^2 = +\tilde{\omega} \wedge \tilde{\theta}^1$$

with $\tilde{\omega} = d\phi + \omega$, where $\phi : U \subset Q \to S^1$ is the angle of the transformation. It follows that $d\omega = d\tilde{\omega}$. Consequently the function

$$K = \frac{d\omega}{\theta^1 \wedge \theta^2} \tag{3}$$

is well-defined, independent of the frame. K is the *curvature of Gauss* and ω is the *Levi-Civita connection*, viewed relative to the frame θ.

A word or two is in order concerning the choice of the form of the equations that define ω. The rotation group $SO(2)$ has for its Lie algebra the space of two-by-two skew symmetric matrices, with typical element

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

Equations 2 read

$$d\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = -\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}. \tag{4}$$

This is the form of *Cartan's structure equations* for the G-structure of a Riemannian surface. The general structure equation has the schematic shape

$$d(\text{coframe}) = (\text{Lie algebra valued one-form}) \wedge (\text{coframe}) + (\text{torsion}).$$

For a Riemannian surface the torsion term is zero.

Exercise 1. *Let ∇ be the Levi-Civita connection associated to our Riemannian metric on the surface. Let (e_1, e_2) be the orthonormal frame dual to the coframe (θ^1, θ^2). Use Cartan's formula $d\theta(X,Y) = X[\theta(Y)] - Y[\theta(X)] - \theta([X,Y])$ to show that*

$$\nabla_X \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = -\begin{pmatrix} 0 & \omega(X) \\ -\omega(X) & 0 \end{pmatrix} \wedge \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Exercise 2. *The Riemannian curvature tensor is defined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. Compute that the Gaussian curvature, as defined in equation 3, satisfies $K = -\langle R(e_1, e_2)e_1, e_2 \rangle$.*

We now redo these computations in global terms on B. First, we observe that equation 1 can also be viewed as defining a *global* $\mathbb{R}^2$-valued one-form on B. Replace θ^1 and θ^2 on the right-hand side by their pull-backs $\pi^*\theta^1$ and $\pi^*\theta^2$ to B. View the left-hand side (the $\tilde{\theta}^i$) as forms Θ^i on B defined at the point whose fiber coordinate is ϕ relative to the local trivialization $B_U \cong U \times G$ (also defined by equation 1). In other words, we rewrite this equation as

$$\begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \pi^*\theta^1 \\ \pi^*\theta^2 \end{pmatrix}.$$

To see that this form is indeed globally defined, let $V \in T_bB$, where b has fiber coordinate ϕ. Then by definition, $\pi^*\theta^i(b)(V) = \theta^i(d\pi_b V)$, so that $\Theta^1(b)(V) = \cos(\phi)\theta^1(d\pi_b V) - \sin(\phi)\theta^2(d\pi_b V)$. Since ϕ is the fiber coordinate for b, the point b represents the basis $(\tilde{\theta}^1, \tilde{\theta}^2)$ for T_p^*Q, $p = \pi(b)$. Think of the coframe b as the linear map $T_pQ \to \mathbb{R}^2$ whose two components are $\tilde{\theta}^1$ and $\tilde{\theta}^2$. With this in mind, we see that $\Theta(b)(V) = b(d\pi_b V)$ as $\mathbb{R}^2$-valued two-forms. This equality shows that the form Θ is indeed globally defined. Θ is called the *tautological*, or *canonical*, one-form. It plays a central role in Cartan's method.

B is three-dimensional, and Θ^1, Θ^2, and $d\phi$ coframe it. It is better to choose Θ^1, Θ^2, and $\alpha = d\phi + \omega$, with ω defined as in equation 2 in terms of a local coframe. This α is indeed globally defined, independent of the choice of frame, even though each of its summands is frame dependent. Equation 4 becomes

$$\begin{pmatrix} d\Theta^1 \\ d\Theta^2 \end{pmatrix} = -\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \wedge \begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix}. \tag{5}$$

This equation in turn uniquely determines α. The one-form α is the *connection one-form associated to the Levi-Civita connection.*

The definition of K in equation 3, together with the fact that $\Theta^1 \wedge \Theta^2 = \pi^*(\theta^1 \wedge \theta^2)$, shows that

$$d\alpha = K\Theta^1 \wedge \Theta^2. \tag{6}$$

Again, this equation can be turned around and taken as a *definition* of K. Equations 5 and 6 are the basic equations of Riemannian surface theory from Cartan's point of view.

Exercise 3. *Using $d^2\Theta^i = 0$, show that equation 5 implies that $d\alpha$ has the form of equation 6, i.e. that it has no $\alpha \wedge \Theta^i$ terms.*

Exercise 4. *Using $d^2\alpha = 0$ and equation 5, show that K as defined by equation 6 must be a function on Q.*