Riemannian geometry.

Immediate Goals: Cartan's structure formulae for Riemannian surfaces, and embedded surfaces. Connections. Curvature. The frame bundle of the homogeneous surfaces as Lie groups.

Longer term goals. Cartan's structure formulae for Riemannian manifolds. Levi-Civita connection and curvature. Laplacian Δ. Space forms.

Starting example. Let $M^2 \subset \mathbb{R}^3$ be an embedded surface. Then the *induced metric* on M^2 is obtained by taking the standard inner product on $\mathbb{R}^3$ and restricting it to the tangent planes $T_m M \subset \mathbb{R}^3$ to the surface. In this way we obtain a smoothly varying inner product on the tangent bundle of M: a Riemannian metric.

Review the definition of a Riemannian metric.

Terminology and notation. The Riemannian metric is variously called the"first fundamental form" (denoted I), the "squared element of arc length" (denoted ds^2) the 'metric tensor' or simply "metric" (denoted g_{ij}). and is also written $\langle \cdot, \cdot \rangle_m$, $m \in M$ to suggest a smoothly varying inner product.

Coordinate version; Gauss' notation: $\vec{x} : U \subset \mathbb{R}^2 \to M^2$. Then $ds^2 = d\vec{x} \cdot d\vec{x}$. Take (u, v) coordinates for the planar domain $U \subset \mathbb{R}^2$ so that (u, v) are coordinates of M. Then Gauss wrote:

$$ds^2 = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2. \tag{1}$$

Thus; $E = \vec{x}_u \cdot \vec{x}_u, F = \vec{x}_u \cdot \vec{x}_v, G = \vec{x}_v \cdot \vec{x}_v$. (Subscripts denote partial derivatives here.) In g_{ij} notation: $g_{11} = E, g_{12} = F = g_{21}, G = g_{22}$.

Exercise 0.1. . *Standard spherical coordinates on $M^2 = S^2$, the unit two-sphere, are $\vec{x}(\theta, \phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$. Compute that $d\vec{x} \cdot d\vec{x} = d\phi^2 + \sin^2(\phi) d\theta^2$*

This is a good time to recall some basic linear algebra: that of quadratic forms. Let $\mathbb{V}$ be a real finite-dimensional vector space. Then there is a canonical bijection between quadratic forms (= homogeneous quadratic polynomials) on $\mathbb{V}$ and bilinear symmetric forms ("inner products") $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$. In one direction this isomorphism sends a bilinear symmetric form $\beta(\cdot, \cdot)$ to the quadratic form $v \mapsto \beta(v, v) := Q_\beta(v)$. In the other direction we use polarization. If Q came from a β (think 'dot product') one can 'polarize' to solve; $\beta(v, w) = \frac{1}{4}(Q(v + w) - Q(v - w))$

If coordinates , ie. a basis $\{e_i\}$ is chosen on $\mathbb{V}$ then both β and Q are given by a symmetric matrix $\beta_{ij} = \beta_{ji}$ where $\beta_{ij} = \beta(e_i, e_j)$. Verify that $Q(v) = \Sigma_{i,j} \beta_{ij} v^i v^j$ with $v = \Sigma v^i e_i$.

Recall : signature, rank.

Recall: β is an inner product if and only if $Q(v) > 0$ whenever $v \neq 0$.

Define the symmetric product $\odot$ of one-forms, say, θ, ν by $\theta \odot \nu = \frac{1}{2}(\theta \otimes \nu + \nu \otimes \theta)$. As a bilinear symmetric form $\theta \odot \nu : (v, w) \mapsto \frac{1}{2}(\theta(v)\nu(w) + \theta(w)\nu(v))$ while as a quadratic form $\theta \odot \nu : v \mapsto \theta(v)\eta(v)$. Then if θ^i is the dual basis to $\{e_i\}$ we have that $\beta = \Sigma \beta_{ij} \theta^i \odot \theta^j$. Note: following completely standard notation we write $du^2, dv^2, dudv$ for $du \odot du, dv \odot dv, du \odot dv$ and with this notation $(du + dv)^2 = du^2 + 2du \odot dv + dv^2$ as it should be.

Graham-Schmidt algorithm = completing the square implies every positive definite inner product β can be written as a sum of squares: $\beta = \Sigma(\theta^i)^2$.

1. Frames and coframes.

The Graham-Schmidt procedure is smooth in the components of the metric. As a consequence, by applying GS to a coordinate frame we can obtain a new frame $\{e_1, e_2\}$, defined on the same coordinate neighborhood , which is everywhere orthonormal :

$$\langle e_1(m), e_1(m) \rangle_m = 1, \langle e_1(m), e_2(m) \rangle_m = 0, \langle e_2(m), e_2(m) \rangle_m = 1$$

Exercise 1.1. *Verify that if we apply GS to the coordinate basis $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ associated to Gauss' form (1) then we get a smooth frame e_1, e_2. (Express e_1, e_2 in terms of E, F, G and $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$. Argue the resulting vector fields are smooth.)*

Exercise 1.2. *Let E_1, E_2 be a frame field and θ^1, θ^2 the dual coframe field. Show that E_1, E_2 is orthonormal if and only if $ds^2 = (\theta^1)^2 + (\theta^2)^2$.*

Suppose now that M is oriented. Then we can insist that our frame and hence our coframe are oriented. When we do so, the *area form* of M is

$$dA = \theta^1 \wedge \theta^2$$

and is globally defined, even though neither θ^1 or θ^2 are globally defined. In this case, we call θ^1, θ^2 an *oriented orthonormal coframe.*

Exercise 1.3. *Let $M \subset \mathbb{R}^3$ be an oriented surface with unit normal vector $\mathbf{N}$ Show that the two-form $\Omega = i_{\mathbf{N}} dx \wedge dy \wedge dz$ restricted to M is the area form dA on M. Hint: show that if $\vec{v}, \vec{w} \in T_m M$ then $\Omega(\vec{v}, \vec{w}) = \vec{N} \cdot (\vec{v} \times \vec{w})$..*

Exercise 1.4. *Suppose that θ^1, θ^2 is an oriented orthonormal coframe for M^2, ds^2 defined on a neighborhood $V \subset M$. Show that the pair of one-forms $\bar{\theta}^1, \bar{\theta}^2$, also defined on U, is also an oriented orthonormal coframe if and only if there is a circle-valued function $\psi : V \to S^1$ such that on V we have that:*

$$\bar{\theta}^1 = \cos(\psi)\theta^1 + \sin(\psi)\theta^2$$
$$\bar{\theta}^2 = -\sin(\psi)\theta^1 + \cos(\psi)\theta^2$$

Corollary 1.5. *The space of oriented orthonormal coframes on a Riemannian surface M, ds^2 forms a circle bundle over M*

2. Structure eqns

Here is a differential forms based algorithm for computing the Gauss curvature K of a Riemannian surface M^2, ds^2.

Step 1. Find an oriented orthonormal coframe θ^1, θ^2: $ds^2 = (\theta^1)^2 + (\theta^2)^2$, $dA = \theta^1 \wedge \theta^2$.

Step 2. Solve the linear equation

$$d\theta^1 = \omega \wedge \theta^2$$
$$d\theta^2 = -\omega \wedge \theta^1$$

. for the one-form ω.

Step 3. Define the function K by

$$d\omega = -K\theta^1 \wedge \theta^2$$

.

The equations of step 2 and 3 are called the *Cartan structure equations* for (M, ds^2).

We have discussed step 1 in detail above.

Regarding step 2.

Exercise 2.1. *Given the coframe θ^1, θ^2 of step 1, show that the eq. of step 2 uniquely determines ω*

This one-form ω is called the "connection one-form" associated to the choice of frame.

Exercise 2.2. *Show that if $\bar{\theta}^1, \bar{\theta}^2$ is another oriented orthonormal coframe then its connection one-form $\bar{\omega}$ is given by $\bar{\omega} = \omega + d\psi$ with ψ as in exercise 1.4.*

Exercise 2.3. *Show that K does not depend on the orientation. If we reverse the orientation of M then K remains unchanged. In particular M does not need to be oriented for the Gaussian curvature to be defined.*

Exercise 2.4. *Compute the Cartan structure equations and find the Gaussian curvature K in the following two cases:*

$$(a) \qquad ds^2 = dr^2 + f(r)^2 d\theta^2$$

$$(b) \qquad ds^2 = \lambda(u,v)^2(du^2 + dv^2)$$

Exercise 2.5. *Compute the curvature for* $ds^2 = \frac{dx^2+dy^2}{y^2}$

Exercise 2.6. *Find f as in (a) of exer 2.4 such that $K = -1$ and f has first order Taylor expansion $f(r) = r + O(r^2)$.*