A DIFFERENTIAL FORMS DERIVATION OF THE THEOREM EGREGIUM: in the guise $K = det(dN)$.

Taken from chapter 7 of Élie Cartan's book: "Les sytèmus différentiels extérieurs et leurs applications géométriques".

We consider a moving orthonormal frame: a "trihedron' defined in a neighborhood of a surface : e_1, e_2, e_3 with e_3 the normal vector N and so e_1, e_2 tangent to the surface. We are surprising the point $A \in M^2 \subset \mathbb{R}^3$ at which the frame is attached: $e_i = e_i(A)$.

Suppose that $A \in \mathbb{R}^3$ represents the point where the frame is attached.

Now differentiate both A and the frame to get: [1]

$$dA = \Sigma\theta^i e_i \tag{1}$$

$$de_i = \Sigma\omega_i^j e_j \tag{2}$$

In the first equation θ^i is the dual coframe [2] to e_i. This, to me, is weird! $\Sigma\theta^i e_i = \Sigma\theta^i \otimes e_i$ is the expansion of the identity matrix. How is it that we "differentiate the moving point" A and get the identity matrix? Think of A as the map: $A \mapsto A$: the identity map of $\mathbb{R}^3$. Its derivative dA is again the identity, but now viewed as a one-form on $\mathbb{R}^3$ with values in $\mathbb{R}^3$ – it is the constant identity-valued one-form!

In class we showed that the ω_j^i of the second equation is skew-symmetric in i and j: $\omega_j^i = -\omega_i^j$ and that this follows immediately from $\langle e_i, e_j\rangle = \delta_{ij} = const.$.

Now differentiate these equations, using $d^2 = 0$, also valid for vector-valued forms like dA. We will use the summation convention over repeated indices without further ado. From $d^2A = 0$ we see that

$$\begin{aligned}
0 &= d(\theta^i e_i) && (3)\\
&= d\theta^i e_i + \theta^i de_i && (4)\\
&= d\theta^i e_i + \theta^i \omega_i^j e_j && (5)\\
&= d\theta^i e_i + \theta^k \omega_k^i e_i && (6)\\
&= (d\theta^i - \omega_k^i \wedge \theta^k) e_k && (7)
\end{aligned}$$

Now since the e_k form a basis we must have

$$d\theta^i = \omega_k^i \wedge \theta^k \tag{8}$$

(*I got stuck at the end of last lecture trying to derive this equation. I tried to prove this a different way.* Continuing to the consequences of $d^2 e_i = 0$ we find

$$\begin{aligned}
0 &= d(\omega_i^j e_j) && (9)\\
&= d\omega_i^j e_j + \omega_i^j de_j && (10)\\
&= d\omega_i^j e_j + \omega_i^j \omega_j^k e_k && (11)\\
&= d\omega_i^k e_k + \omega_i^j \omega_j^k e_k && (12)\\
&= (d\omega_i^k + \omega_i^j \omega_j^k) e_k && (13)
\end{aligned}$$

so that

$$d\omega_i^k = -\omega_i^j \wedge \omega_j^k \tag{14}$$

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[1] Cartan viewed such a trihedron as being intimately linked to the group of Euclidean motions Then you can think of $0 \mapsto A$ as a translation. We can think of e_1, e_2, e_3 as being the columns of a matrix which describes how to rotate the standard xyz axes attached to 0 to axes parallel to $e_1(A), e_2(A), e_3(A)$ now attached at A.

[2] so that $\theta^i(A)(V) = \langle e_i(A), v\rangle$.

Now let $f : M^2 \to \mathbb{R}^3$ be the inclusion. We pull the structure equations back to the surface M using f. The main consequence of pull back is that $f^*\theta^3 = 0 = f^*d\theta^3$. We find

$$d\theta^1 = \omega^1_2 \wedge \theta^2 \tag{15}$$

$$d\theta^2 = \omega^2_1 \wedge \theta^1 \tag{16}$$

$$0 = \omega^3_1 \wedge \theta^1 + \omega^3_2 \wedge \theta^2 \tag{17}$$

$$\tag{18}$$

and

$$d\omega^1_2 = -\omega^3_2 \wedge \omega^1_3 \tag{19}$$

$$\tag{20}$$

Comparing eqns (15) and (16) with the earlier set of eqns $d\theta^1 = \omega \wedge \theta^2$ and $d\theta^2 = -\omega \wedge \theta^1$ we see that

$$\omega^1_2 = \omega \tag{21}$$

Use this fact and eq (18) compared to $d\omega = -K\theta^1 \wedge \theta^2$ to get that

$$\begin{aligned} K &= d\omega^1_2(e_2, e_1) && (22)\\ &= -\omega^3_2 \wedge \omega^1_3(e_2, e_1) && (23)\\ &= \omega^3_2 \wedge \omega^1_3(e_1, e_2) && (24)\\ &= \omega^3_2(e_1)\omega^1_3(e_2) - \omega^3_2(e_2)\omega^1_3(e_1) && (25)\\ &= det(M) && (26) \end{aligned}$$

where

$$M = \begin{pmatrix} \omega^3_2(e_1) & \omega^1_3(e_1) \\ \omega^3_2(e_2) & \omega^1_3(e_2) \end{pmatrix}$$

We will be done once we relate M to dN. Since $N = e_3$ we have from eq (2) that

$$dN = \omega^1_3 e_1 + \omega^2_3 e_2 \tag{27}$$

Hence the matrix of dN relative to the e_1, e_2 basis is

$$dN = \begin{pmatrix} \omega^1_3(e_1) & \omega^1_3(e_2) \\ \omega^2_3(e_1) & \omega^2_3(e_2) \end{pmatrix}$$

Using $\omega^i_j = -\omega^j_i$ we see that indeed the determinant of dN equals that of M.

QED