§5. ORIENTED MANIFOLDS

IN ORDER to define the degree as an integer (rather than an integer modulo 2) we must introduce orientations.

DEFINITIONS. An orientation for a finite dimensional real vector space is an equivalence class of ordered bases as follows: the ordered basis $(b_1, \cdots, b_n)$ determines the *same orientation* as the basis $(b'_1, \cdots, b'_n)$ if $b'_i = \sum a_{ij} b_j$ with $\det(a_{ij}) > 0$. It determines the *opposite orientation* if $\det(a_{ij}) < 0$. Thus each positive dimensional vector space has precisely two orientations. The vector space R^n has a *standard* orientation corresponding to the basis $(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$.

In the case of the zero dimensional vector space it is convenient to define an "orientation" as the symbol $+1$ or -1.

An *oriented* smooth manifold consists of a manifold M together with a choice of orientation for each tangent space TM_x. If $m \geq 1$, these are required to fit together as follows: For each point of M there should exist a neighborhood $U \subset M$ and a diffeomorphism h mapping U onto an open subset of R^m or H^m which is *orientation preserving*, in the sense that for each $x \in U$ the isomorphism dh_x carries the specified orientation for TM_x into the standard orientation for R^m.

If M is connected and orientable, then it has precisely two orientations.

If M has a boundary, we can distinguish three kinds of vectors in the tangent space TM_x at a boundary point:

1) there are the vectors tangent to the boundary, forming an $(m - 1)$-dimensional subspace $T(\partial M)_x \subset TM_x$;

2) there are the "outward" vectors, forming an open half space bounded by $T(\partial M)_x$;

3) there are the "inward" vectors forming a complementary half space.

Each orientation for M determines an orientation for ∂M as follows: For $x \in \partial M$ choose a positively oriented basis $(v_1, v_2, \cdots, v_m)$ for TM_x in such a way that $v_2, \cdots, v_m$ are tangent to the boundary (assuming that $m \geq 2$) and that v_1 is an "outward" vector. Then $(v_2, \cdots, v_m)$ determines the required orientation for ∂M at x.

If the dimension of M is 1, then each boundary point x is assigned the orientation -1 or $+1$ according as a positively oriented vector at x points inward or outward. (See Figure 8.)

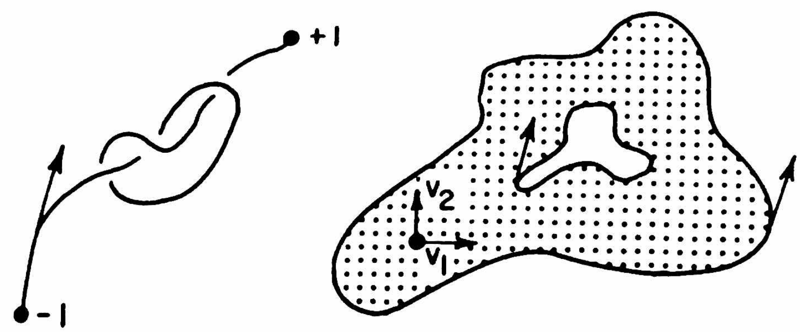

Figure 8. How to orient a boundary

As an example the unit sphere $S^{m-1} \subset R^m$ can be oriented as the boundary of the disk D^m.

THE BROUWER DEGREE

Now let M and N be oriented n-dimensional manifolds without boundary and let

$$f : M \to N$$

be a smooth map. If M is compact and N is connected, then the degree of f is defined as follows:

Let $x \in M$ be a regular point of f, so that $df_x : TM_x \to TN_{f(x)}$ is a linear isomorphism between oriented vector spaces. Define the *sign* of df_x to be $+1$ or -1 according as df_x preserves or reverses orientation. For any regular value $y \in N$ define

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} df_x .$$

As in §1, this integer $\deg(f; y)$ is a locally constant function of y. It is defined on a dense open subset of N.

Theorem A. *The integer* $\deg(f; y)$ *does not depend on the choice of regular value* y.

It will be called the *degree* of f (denoted $\deg f$).

Theorem B. *If* f *is smoothly homotopic to* g, *then* $\deg f = \deg g$.

The proof will be essentially the same as that in §4. It is only necessary to keep careful control of orientations.

First consider the following situation: Suppose that M is the boundary of a compact oriented manifold X and that M is oriented as the boundary of X.

Lemma 1. *If* $f : M \to N$ *extends to a smooth map* $F : X \to N$, *then* $\deg(f; y) = 0$ *for every regular value* y.

Proof. First suppose that y is a regular value for F, as well as for $f = F \mid M$. The compact 1-manifold $F^{-1}(y)$ is a finite union of arcs and circles, with only the boundary points of the arcs lying on $M = \partial X$. Let $A \subset F^{-1}(y)$ be one of these arcs, with $\partial A = \{a\} \cup \{b\}$. We will show that

$$\text{sign } df_a + \text{sign } df_b = 0,$$

and hence (summing over all such arcs) that $\deg(f\,; y) = 0$.

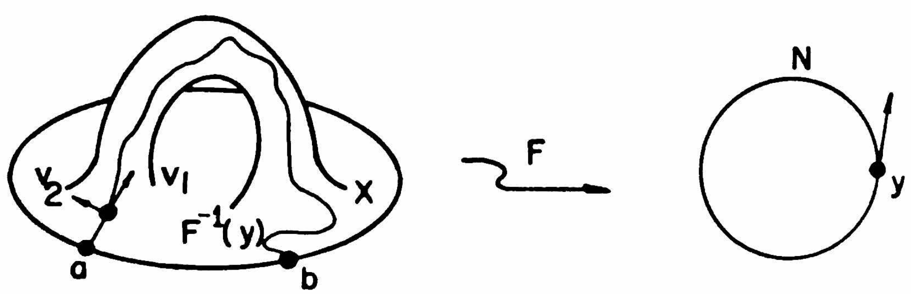

Figure 9. How to orient $F^{-1}(y)$

The orientations for X and N determine an orientation for A as follows: Given $x \in A$, let $(v_1, \cdots, v_{n+1})$ be a positively oriented basis for TX_x with v_1 tangent to A. *Then* v_1 *determines the required orientation for* TA_x *if and only if* dF_x *carries* $(v_2, \cdots, v_{n+1})$ *into a positively oriented basis for* TN_y.

Let $v_1(x)$ denote the positively oriented unit vector tangent to A at x. Clearly v_1 is a smooth function, and $v_1(x)$ points outward at one boundary point (say b) and inward at the other boundary point a.