

For next time.

A) Volunteer presentation.

$$\Lambda^2 W, \Lambda^2 W^*$$

B) any of the HW.

$$C) \mathbb{C}P^1 = S^2.$$

forms on  $\mathbb{R}^n$ .

1<sup>st</sup> exterior algebra.  
an  $\mathbb{R}$ -algebra.

Basis  $1, dx^1, \dots, dx^n$ .

Relations  $dx^i \wedge dx^j = -dx^j \wedge dx^i$

Exer  $(dx^1 \wedge dx^2 + dx^3 \wedge dx^4)^2 = 2dx^1 dx^2 dx^3 dx^4$

(Fl. writes:  $dx^i dx^j = -dx^j dx^i$  etc.)

Notation  $\Lambda^k \mathbb{R}^n$ , better:  $\Lambda^k(\mathbb{R}^{n \times 1})$

$K$ -forms ~~at a pt~~:  $\Lambda^k \mathbb{R}^n$ .

sums of order  $k$  monomials in  
the  $dx^i$ .

Interpretation:  $\Lambda^k \mathbb{R}^{n \times 1}$

= completely antisym.  
 $K$ -linear fns

$\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_k \longrightarrow \mathbb{R}$ .

eg:  $dx^i$  dual to  $e_i = \frac{\partial}{\partial x^i} = (0, \dots, \underset{i\text{th}}{1}, \dots, 0)$

~~$dx^i dx^j$~~

so:

$$dx^i(v) = v^i \quad \text{if}$$

$$v = \sum v^j \frac{\partial}{\partial x^j}$$

$$(dx^i dx^j)(v, w) = dx^i(v) dx^j(w) - dx^i(w) dx^j(v) \\ = v^i w^j - w^i v^j$$

= signed area of proj  
of  $v, w$  onto

~~$\frac{\partial}{\partial x^i}$~~ ,  $\frac{\partial}{\partial x^j}$  plane.

$S \sim \mathbb{R}^2$ :

$dx^1 dx^2 \leftrightarrow$  "det"

$\sim \mathbb{R}^3$

$dx^1 dx^2 dx^3 \leftrightarrow$  "det" etc.

Then a  $k$ -form on  $\mathbb{R}^n$   
is a smooth map

$$\mathbb{R}^n \longrightarrow \Lambda^k \mathbb{R}^n,$$

in coord

$$1\text{-form: } \sum a_i(x) dx^i$$

$$x = (x^1, \dots, x^n)$$

$$2\text{-form } \sum a_{ij}(x) dx^i dx^j$$

sum: either  $i < j$

or:

$$a_{ij} = -a_{ji} \leftarrow \text{Cartan prefers}$$

$$3\text{-form } \sum a_{ijk}(x) dx^i dx^j dx^k$$

$$\text{either } a_{ijk} = a_{kij} = a_{jki} \\ = -a_{jik} \dots$$

or insist

$$i < j < k$$

$\frac{1}{3!}$  relaty to 2-ways

On an manifold,

cover  $x^i$

same deal.

locally  $\sum a_I dx^I$

globally?

$$x \quad u^i = u^i(x^1, \dots, x^n)$$

inverse

$$x^i = x^i(u^1, \dots, u^n).$$

$$dx^i = \sum \frac{\partial x^i}{\partial u^j} du^j$$

&

$$du^j = \sum \frac{\partial u^j}{\partial x^i} dx^i$$

indices change of cover  $u$  to  
from  $\Delta$  show how to  
express the in another  
basis cover system.

Vectors:

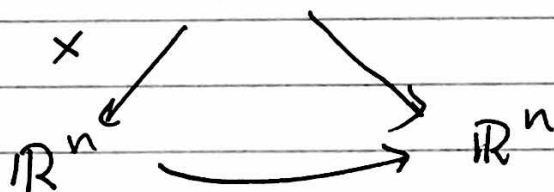
$$v \in T_x M.$$

$$v = \sum v^i \frac{\partial}{\partial x^i} = \sum \tilde{v}^j \frac{\partial}{\partial u^j}$$

$$= \sum v^i \frac{\partial}{\partial x^i} = \sum \tilde{v}^j \frac{\partial}{\partial u^j}$$

$$\Rightarrow \boxed{\tilde{v}^j = \sum \frac{\partial u^j}{\partial x^i} v^i}$$

$W, U \subset M$



$$u = u(x),$$

$$= \phi_u \circ \phi_x^{-1}(x)$$

$$= \psi(u)$$

"contravariant"

$$\tilde{v} = d\psi_x v. \quad \text{rep.}$$

Forms. Exer show if

$$p = \sum p_i dx^i = \sum \tilde{p}_j du^j$$

$$\text{Then } \tilde{p} = (d\psi_x^{-1})^+ p.$$

"covariant"

verify it.

$$\begin{aligned}\sum p_i dx^i &= \sum p_i \frac{\partial x^i}{\partial u^j} du^j \\ &= \sum \tilde{p}_i du^j\end{aligned}$$

$$\overset{\text{so}}{\tilde{p}_i} = \sum p_i \frac{\partial x^i}{\partial u^j}$$

$$\text{But: } \left( \frac{\partial u^j}{\partial x^i} \right)^{-1} = \frac{\partial x^i}{\partial u^j}$$

& look at sum: now on  
top index  
not bottom.