¿Why $\vec{F}\cdot\vec{n}\,dA = F_1\,dy\wedge dz + F_2\,dz\wedge dx + F_3\,dx\wedge dy$?

Take a small piece of a surface $S \subset \mathbb{R}^3$

Parameterize it by $(u,v) \mapsto \vec{x}(u,v) \in \mathbb{R}^3$

Then $\vec{n} = \dfrac{\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}}{\left\| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right\|}$. $\left\{ \begin{array}{l} \text{so} \\ \mathbb{R}^2 \xrightarrow{\vec{x}} \mathbb{R}^3 \\ \operatorname{im}\vec{x} \subset S, \end{array} \right.$

Since $\vec{x}_u := \frac{\partial \vec{x}}{\partial u}$, $\vec{x}_v := \frac{\partial \vec{x}}{\partial v}$ are lin. ind. vector fields tangent to S. Also

"dA" $= \|\vec{x}_u \times \vec{x}_v\|\,du\,dv$.

(see a book on vector calc.).

So $\vec{n}\,dA = \vec{x}_u \times \vec{x}_v\,du\,dv$

$= \vec{x}_u\,du \times \vec{x}_v\,dv.$

So $\vec{F}\cdot\vec{n}\,dA = (\vec{F}\cdot\vec{x}_u \times \vec{x}_v)\,du\wedge dv.$

By

(left from Sam)

2/

But.

$\cancel{F_1 dx \wedge y \wedge d}$ $F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$

$= \vec{F} \lrcorner \, dx \wedge dy \wedge dz$

So that, if $\vec{v}_1, \vec{v}_2 \parallel S$

are tgt vectors for S we

have

$(F_1 dy \wedge dz + \cdots F_3 dx \wedge dy)(\vec{v}_1, \vec{v}_2)$

$= dx \wedge dy \wedge dz(\vec{F}, \vec{v}_1, \vec{v}_2)$

But $dx \wedge dy \wedge dz(\vec{v}_3, \vec{v}_1, \vec{v}_2)$

$= dx \wedge dy \wedge dz(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

$= \det[\vec{v}_1 \downarrow, v_2 \downarrow, v_3 \downarrow]$

$= \vec{v}_3 \cdot (\vec{v}_1 \times \vec{v}_2).$

3/

~~Ba~~ Let $i: S \hookrightarrow \mathbb{R}^3$

be the inclusion.

The equality "$\vec{F}\cdot\vec{n}\,dA = F_1\, dy\wedge dz + \cdots$"

Means

$$\vec{F}\cdot\vec{n}\,dA = i^*(F_1\, dy\wedge dz + F_2\, dz\wedge dx + F_3\, dx\wedge dy)$$

$$= i^*\, \vec{F} \lrcorner\, dx\wedge dy\wedge dz.$$

Now. $\vec{X}_u, \vec{X}_v$ are a basis for

TS

We have just seen:

$$(\vec{F}\cdot\vec{n}\,dA)\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \vec{F}\cdot(\vec{X}_u \times \vec{X}_v)$$

$$\& \quad (i^*\vec{F}\lrcorner\, dx\wedge dy\wedge dz)\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$$

$$= dx\wedge dy\wedge dz\left(\vec{F}, \frac{\partial \vec{X}}{\partial u}, \frac{\partial \vec{X}}{\partial v}\right)$$

$$= \vec{F}\cdot(\vec{X}_u \times \vec{X}_v)$$

So: the two 2-forms are equal.