Stokes, & $\int_\Sigma \omega$

Following G-P 4.4. sec'n at first.

$U \subset \mathbb{R}^K$, U bdd.
$a : \overline{U} \to \mathbb{R}$

$$\int_{\substack{U \\ \shortparallel \\ F(V)}} a\, dx^1 dx^2 \cdots dx^K = \int_V (F^* a) |\det(dF)| dy^1 \cdots dy^K$$

~~for~~ where $F : V \to U$ a diffeo

in coord:
$F^i(y^1, \ldots, y^K) = x^i(y^1, \ldots, y^K)$
$\det dF = \dfrac{\partial x^i}{\partial y^j}$ as a matrix

So if F is orient preserving.

$$\int_{F(V)} a\, dx^1 \cdots dx^K = \int_V (a \circ F) \det dF\, dy^1 \cdots dy^K$$

Now if $\omega = a\, dx^1 \wedge \cdots \wedge dx^K$
we set
$$\int_U \omega = \int_U a\, dx^1 \cdots dx^K$$

& formula reads
$$\int_U \omega = \int_V F^* \omega$$

param. indep.

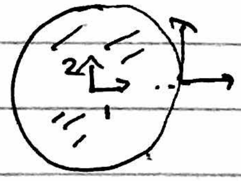
if F reverses orientation

$$\int \omega = -\int_{U} F^{*}\omega$$

So: we $\int$ forms over oriented objects

An orientation of a domain or a submfd induces one of its ∂

n ⇑ outer normal "outward pointing

eg: 

std orient disc D
$\to$ counterclockwise orient. of $\partial D = S^1$

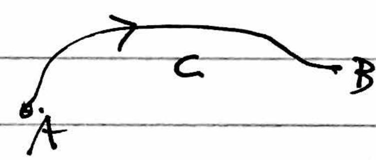
Stokes Thm

$\omega \in \Omega^k(M)$, so $d\omega \in \Omega^{k+1}(M)$

$\Sigma \subset M$ an oriented ~~k-dim'l.~~ submfd $k+1$ dim.

$\partial\Sigma$ its ∂ w induced orientation.

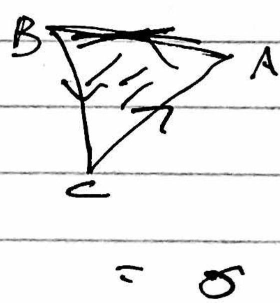
Then
$$\int_{\partial\Sigma}\omega = \int_{\Sigma} d\omega.$$

Case $k=0$; $\Sigma =$ 

$$\int_{\partial c} f = \int_c df.$$

or
$$f(B)-f(A) = \int_c df.$$

Also holds for chains or cube complexes on.

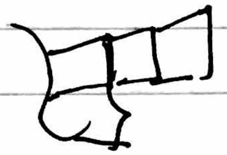
eg. ~~ch~~ 2-chain 

$\xrightarrow{\sigma} M.$

$$\partial\sigma = \sigma|_{\partial\,\text{simplex}} = \sigma_{AB} + \sigma_{BC} + \sigma_{CA}$$

Sketch proof:

"Cubulate Σ"



By additivity enough to prove on one cube.

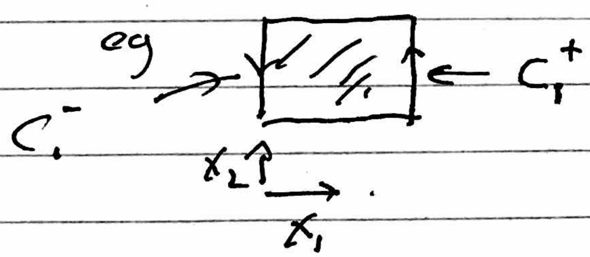
so $\omega \in \Omega^k(\mathbb{R}^{k+1})$

$\Sigma =$ unit cube. $0 \le x_i \le 1$

$\partial \Sigma =$ union of 2^k faces $x_i = 0$ or $x_i = 1$.

$= \sum_i C_i^+ - C_i^-$

eg



$$\omega = \omega_1\, dx^2 \wedge dx^3 \wedge \dots \wedge dx^n + \omega_2\, dx^1 \wedge dx^3 \wedge \dots \wedge dx^n$$

$$\therefore d\omega = \frac{\partial \omega_1}{\partial x^1} dx^1 \wedge dx^2 \dots dx^n + \frac{\partial \omega_2}{\partial x^2} dx^1 dx^2 \dots dx^n$$

$$\int_{\partial \Sigma} \omega = \sum \int_{C_i^+} \omega - \int_{C_i^-} \omega.$$

~~Lets~~ f

Now: $i^*\omega \Big|_{C_i^+} = \omega_i dx^1 \wedge \ldots \cancel{dx^i} \ldots dx^n$

since $dx^i \equiv 0$ on $C_i^\pm$

so $\int_{\partial\Sigma} \omega = \sum \int_{x_i=1} \omega_i dx^1 \ldots \cancel{dx^i} dx^n - \int \omega_i \cdot dx^1$

Focus on $i=1$ case.

$\int_{C_1^+} \omega_1(1, x_2, \ldots, x_n) dx^2 \ldots dx^n - \int \omega_1(0, x_2 \ldots x_n) dx^2 \ldots dx^n$

$= \int_I \int_0^1 \left(\frac{\partial \omega_1}{\partial x^1} dx^1 \right) dx^2 \ldots dx^n$

$= \int_\Sigma \frac{\partial \omega_1}{\partial x^1} dx^1 \ldots dx^n$