L_X.

X v-fld.

ϕ_t or ϕ_t^X or $\exp(tX)$

its flow.

eg: $X = \frac{\partial}{\partial x^1}$ on $\mathbb{R}^n$.

what is ϕ_t?

τ: any tensor field;

so: metric, 1-form, k-form, vector field, function.

or $\tau = \sum \tau_{i_1 \ldots i_k}^{j_1 \ldots j_\ell} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \frac{\partial}{\partial x^{j_\ell}}$

$\phi_t^* \tau$ makes sense.
is another tensor field.

$$\boxed{L_X \tau = \frac{d}{dt}\Big|_{t=0} \phi_t^* \tau}$$

L_X 2

Cartan's Magic formula.

$$L_X = d i_X + i_X d.$$

on $\Omega(M)$.

<u>Pf</u> step 1:

on 0 forms:

$$L_X f = df(X) = i_X df. \checkmark$$

since $df(X) = \frac{d}{dt}\Big|_{t=0} f \circ \phi_t$.

<u>step 2</u>: $\Omega(M)$ is an alg. over the ring $C^\infty(M)$.

~~B Both L~~ Show Both L_X & $d i_X + i_X d$ are derivations of $\Omega(M)$

(degree 0)

'Derivation' $\qquad L_X$ 3

<u>Means</u>:

$$\delta(\alpha \wedge \beta) = \delta\alpha \wedge \beta + \alpha \wedge \delta\beta.$$

$$\cancel{\&} \quad \alpha \in \Omega^k(M) \Rightarrow \delta\alpha \in \Omega^k(M)$$

Why? For L_X:

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^*\alpha \wedge \phi_t^*\beta.$$

Now differentiate!

$$\Rightarrow L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta.$$

<u>In</u> <u>uses</u> in

~~Any~~ module the derivative of an automorphism ~~m~~ is a ~~d~~ derivation

For $\delta = d i_X + i_X d$.

Pf that δ is a derivation: d is a degree -1 "~~sp~~ superderivation"

i_X = degree $+1$ superderivation

Means:

$$D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge D\beta.$$

"super part"

verify directly.

<u>Next</u> algebra:

If $D_1 : A \longrightarrow A$ deg -1 "superderivation"

$D_2 : A \to A$ deg $+1$ superderiv.

then $[D_1, D_2]_+ := D_1 D_2 + D_2 D_1$

is a degree 0 derivation.

<u>Step 2</u> Both L_X
& $\delta := d i_X + i_X d$.

Commute w/ d:

For L_X: $\phi_t^* d = d \phi_t^*$.

Now take $\frac{d}{dt}\Big|_{t=0}$.

For δ:

$$\begin{aligned} \delta d &= (d i_X + i_X d) d \\ &= d i_X d + i_X d^2 \\ &= d i_X d. \\ d \delta &= d(d i_X + i_X d) \\ &= d^2 i_X + d i_X d \\ &= d i_X d \ \checkmark. \end{aligned}$$

Result follows:

Locally any k-form looks like. $\sum f_I dx^I$; $\{ dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$

~~A~~ By induc. on $\underline{\underline{k}}$.

$k = 0$ ✓.

$k = 1$: $\alpha = \sum f_i dx^i$

$$L_X \alpha = \sum L_X (f_i dx^i)$$

$$= \cancel{\sum b}$$

~~Ex~~ similarly.

$$\delta \alpha = \sum \delta (f_i dx^i)$$

$$L_X f dg = L_X f \, dg + f L_X dg$$

$$= L_X f \, dg + f d L_X g.$$

$$= X[f] dg + f d(X[g])$$

$$\delta(f dg) = (\delta f) dg + f d \delta g.$$

Ex.