Gauss-Bonnet. Poincaré-Hopf. Degree
The Gauss-Bonnet theorem.

Theorem 0.1 (G-B)**.** *Let M be a compact Riemannian surface, either without boundary, or whose boundary is the a union of geodesics. Then the integral of the curvature over M equals the Euler-characteristic of M:*

$$\int_M K dA = \chi(M).$$

The theorem asserts that a local geometric invariant (integrated) equals a global topological invariant. The theorem is a model for a huge swath of work in differential geometry in the 20th century, culminating with the Atiyah-Singer index theorem.

Proof. I will prove G-B in the case of an oriented surface M without boundary. Begin by choosing a vector field v on M with only isolated zeros. Normalize v to get a unit vector field $e_1 = v/|v|$ defined away from the zeros of v. Delete a tiny disc $D_\epsilon(p)$ around each of the zero p of v to obtain the manifold *with boundary*

$$M_\epsilon = M \setminus \bigcup_{i=1}^{N} D_\epsilon(p_i)$$

Here the p_i are the zeros of v, listed out. Use the orientation of M to obtain the orthogonal unit length vector e_2. Let θ^1, θ^2 be the dual coframe, defined on M_ϵ. Form the connection form ω via the structure equations, so that

$$d\theta^1 = \omega \wedge \theta^2$$

and

$$d\omega = -K\theta^1 \wedge \theta^2$$

Now the area form is $dA = \theta^1 \wedge \theta^2$. Apply Stokes':

$$\int_{M_\epsilon} K dA = -\int_{M_\epsilon} d\omega = \Sigma_{i=1}^{N} \int_{C_\epsilon(p_i)} \omega$$

where the $C_\epsilon(p_i)$ are the boundaries of $D_\epsilon(p_i)$ and are small circles about p_i. In performing the integration about the circles we orient the circles in their usual "counterclockwise" orientation which they receive as boundaries of the small discs. This orientation is opposite to the orientation they receive as forming the boundary of M_ϵ, hence the switch in sign in the last equation. See figure for orientation.

We will show:

$$\int_{C_\epsilon(p_i)} \omega = deg(v; p_i)$$

where "$deg(v;p)$" is an integer called the degree: here the degree of an isolated zero p of a vector field. Once we know this fact, we invoke the theorem of Poincaré-Hopf

Theorem 0.2. *If v is a vector field with isolated zeros on a compact manifold, and M is either without boundary, or with boundary but such that v is everywhere pointing out of the boundary, then*

$$\chi(M) = \Sigma_{\{p: v(p)=0\}} ind(v; p).$$