Let (x, y) be standard coordinates on the plane $\mathbb{R}^2$.

Definition 0.1 *A one-form on the plane is a a linear expression in the indeterminates dx and dy with coefficents smooth functions. Thus a one-form is an expression of the form*

$$\alpha := F_1(x, y)dx + F_2(x, y)dy$$

.

A two-form on the surface is a linear expression in the indeterminate $dx \wedge dy$ with coefficent a smooth function. Thus a two-form has the shape

$$\omega = G(x, y)dx \wedge dy.$$

Integrating one- forms. One-forms are the integrands for line integrals. If $c(t) = (x(t), y(t))$, $a \leq t \leq b$ is a smooth curve in the plane and α is a one-form as above,then

$$\int_c \alpha = \int_a^b F_1(x(t), y(t))\frac{dx}{dt}dt + F_2(x(t), y(t))\frac{dy}{dt}dt$$

and this integral is termed the integral of the one-form α over the curve c.

The differential.

Definition 0.2 *If $f = f(x, y)$ is a smooth function on the plane then its differential is the one-form*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

The fundamental theorem of calculus reads, in the language of forms:

$$\int_c df = f(c(b)) - f(c(a)).$$

Exercise 1 . *Verify that the integral of a one-form is independent of coordinates. Thus, if (u, v) are another good choice of coordinates on the plane and the curve c is defined in the overlap of the (u, v) and (x, y) coordinate chart, show that $\int_c \alpha$ is the same, regardless of what chart you use for the computation. Use the obvious transformation rule $dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$ and similarly for dy to transform between forms written in different charts.*

The wedge product. The basic rule regarding the wedge product is that $df \wedge dg = -dg \wedge df$ where f and g are functions. Thus $dx \wedge dx = dy \wedge dy = 0$. We posit that $dx \wedge dy \neq 0$ so that, as the definition above asserts, ,pointwise, the space of two-forms have $dx \wedge dy$ as a basis.

We extend d to a map from one-forms to two forms by the two laws

$$d^2 = 0$$

and

$$d(f\theta) = df \wedge \theta + fd\theta.$$

Exercise 2 $d(Pdx + Qdy) = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy$.

Exercise 3 *Use the previous exercise to verify that indeed* $ddf = 0$ *for any smooth function* f *on the plane.*

Integrating two-forms We integrate a two-form over bounded domain Ω in the plane by changing $dx \wedge dy$ to the Lebesgue integrand $dxdy$. Thus we define $\int_\Omega \omega = \int\int_\Omega G(x,y)dxdy$ with $\omega = G(x,y)dx \wedge dy$ as above.

Exercise 4 . *Take* $\alpha = Pdx + Qdy$ *as per exercise 2.*

Verify that Green's theorem in the plane reads

$$\int_\Omega d\alpha = \int_c \alpha$$

where Ω *is any bounded domain in the plane whose boundary is a smooth closed curve* c *(Think of the unit disc whose boundary is the unit circle.)*

How must c *be oriented?*

Looking ahead: The formula

$$\int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha$$

is called Stoke's theorem and is valid for integrating smooth forms α of any degree k. The differential $d\alpha$ of a k-form is a $k+1$-form so the object over which it is to be integrated, Σ, has dimension $k+1$ and must be "oriented". The boundary $\partial\Sigma$ of this object is an oriented k-dimensional object.

Change of variables in integration Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a change of coordinates $\Phi(u,v) = (x(u,v), y(u,v))$. We define the pull-back of forms by substituting variables and expanding out in the obvious way. Algebraically then, for a function f we have $(\Phi^* f)(u,v) = f(x(u,v), y(u,v))$ while for the basis one-forms we have $\Phi^* dx = d\Phi^* x = d(x(u,v)) = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$ and similarly $\Phi^* dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$. Finally $\Phi^*\alpha \wedge \beta = \Phi^*\alpha \wedge \Phi^*\beta$ and $\Phi^* f\alpha = \Phi^* f \Phi^*\alpha$.

Exercise 5 . $\Phi^* dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)} du \wedge dv$ *where* $\frac{\partial(x,y)}{\partial(u,v)}$ *is the Jacobian of the map* $(u,v) \mapsto \Phi(x,y) = (u(x,y), v(x,y))$, *which is to say, the determinant of the two-by-two derivative matrix of this transformation.*

Exercise 6 *Verify the change of variables formula for two-forms:* $\int_\Omega \omega = \int_{\Phi^{-1}(\Omega)} \Phi^*\omega$.

Exercise 7 *Verify the change of variables formula for one-forms:* $\int_c \alpha = \int_{\Phi^{-1}c} \Phi^*\alpha$ *where* $\Phi^{-1}c$ *is the curve in the u-v plane parameterized as* $t \mapsto \Phi^{-1} \circ c(t)$

Exercise 8 *Verify that line integrals are independent of parameterization: Let* $\tilde{c}(\tau) = c(t(\tau))$ *where* $\tau \mapsto t(\tau)$ *is smooth strictly monotone change of variables mapping the interval* $[A,B]$ *to the interval* $[a,b]$ *so that the new curve* $\tilde{c}$ *is parameterized by* $[A,B]$ *Show that change of variables formula for one-forms:* $\int_{\tilde{c}} \alpha = \int_c \alpha$

Summarizing :the integrals of one and two-forms is independent of coordinates and of parameterization.