Let (u, v) be local coordinates on some smooth surface Σ.

Definition 0.1 (Preliminary) *A one-form on a surface is a a linear expression in the indeterminates du and dv with coefficents smooth functions. Thus a one-form is an expression of the form*

$$\alpha := F_1(u,v)du + F_2(u,v)dv$$

.

A two-form on the surface is a linear expression in the indeterminate $du \wedge dv$ with coefficent a smooth function. Thus a two-form has the shape

$$\omega = G(u,v) du \wedge dv.$$

Integrating one- forms. One-forms are the integrands for line integrals. In local coordinates a curve on the surface is given by $c(t) = (u(t), v(t))$, $a \leq t \leq b$. Then, with α as above,

$$\int_c \alpha = \int_a^b F_1(u(t), v(t))\frac{du}{dt}dt + F_2(u(t), v(t))\frac{dv}{dt}dt$$

and this integral is termed the integral of the one-form α over the curve c.

The differential.

Definition 0.2 *Let $f = f(u,v)$ be a function on the surface Σ. The differential of f is the one-form*

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv.$$

The fundamental theorem of calculus reads, in the language of forms:

$$\int_c df = f(c(b)) - f(c(a)).$$

Exercise 1 . *Verify that the integral of a one-form is independent of coordinates. Thus, if (x, y) are other coordinates on the surface and if the curve c is defined in the overlap of the (u, v) and (x, y) coordinate chart, show that $\int_c \alpha$ is the same, regardless of what chart you use for the computation. Use the obvious transformation rule $dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$ and similarly for dy to transform between forms written in different charts.*

The wedge product. The basic rule regarding the wedge product is that $df \wedge dg = -dg \wedge df$ where f and g are functions. Thus $du \wedge du = dv \wedge dv = 0$. We posit that $du \wedge dv \neq 0$ so that, as the definition above asserts, ,pointwise, the space of two-forms have $du \wedge dv$ as a basis.

We extend d to a map from one-forms to two forms by the two laws

$$d^2 = 0$$

and

$$d(f\theta) = df \wedge \theta + fd\theta.$$

Exercise 2 $d(F_1du + F_2dv) = (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})du \wedge dv$.

Exercise 3 *Assuming the validity of the previous exercise, and the properties of the wedge product, use definition 0.2 to show that* $ddf = 0$ *for any smooth function* f*.*

Integrating two-forms We can integrate two-forms over parameterized regions in the surface. If $\Phi : \Omega \subset \mathbb{R}^2 \to \Sigma$ is such a parameterization, ie smooth map expressed in coordinates as $\Phi(x, y) = (u(x, y), v(x, y))$ and if $\omega = F(u, v)du \wedge dv$ is a two-form, we define

$$\int_{\Phi(\Omega)} \omega = \int_{\Omega} \Phi^* \omega$$

where

$$\Phi^* \omega = F(u(x, y), v(x, y))\Phi^* du \wedge dv$$

and where $\Phi^* du \wedge dv$ means to express u, v as functions of x, y according to Φ , to take their differentials and to wedge the result.

Exercise 4 . $\Phi^* du \wedge dv = \frac{\partial(u,v)}{\partial(x,y)} dx \wedge dy$ *where* $\frac{\partial(u,v)}{\partial(x,y)}$ *is the Jacobian of the map* $(x, y) \mapsto \Phi(x, y) = (u(x, y), v(x, y))$*, which is to say, the determinant of the two-by-two derivative matrix of this transformation.*

Theorem 1 . *Let* Ω *be an (oriented) region on* Σ *whose boundary* $\partial\Omega$ *is the closed curve* c*. Then*

$$\int_{\Omega} d\alpha = \int_c \alpha$$

Exercise 5 . *Take* $\Sigma = \mathbb{R}^2$ *with coordinates* x, y*. Write* $\alpha = Pdx + Qdy$*. Verify that the above theorem coincides with Green's theorem in the plane.*

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Coordinate invariance; how forms change under change of coordinates. Redux. Recall exercise 1: Let x, y be another pair of coordinates for our surface. Thus the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is invertible whereever both coordinates are defined. To express the one-form $\alpha = F_1du + F_2dv$ in terms of the new basis dx, dy, we write $\alpha = \tilde{F}_1dx + \tilde{F}_2dy$, and view of x, y as functions of u, v to compute the change of basis formula relating dx, dy to du, dv.

The Jacobian factor in exercise 4 is also precisely the correction factor needed to make the change of variables formula for multi-variable calculus work for two-dimensional integrals.

To summarize

Proposition 1 *The integral of a 2-form over a surface, or of a 1-form over a curve can be done in coordinates but the result is independent of the coordinates used to express the forms.*