Math 209 Riemannian Geometry

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Problem. Let $M^2 \subset \mathbb{R}^3$ be embedded surface. Then the induced metric on M^2 is obtained by taking the standard inner product on $\mathbb{R}^3$ and restricting it to the tangent planes $T_m M \subset \mathbb{R}^3$ to the surface. In this way we obtain a smoothly varying inner product on the tangent bundle of M: a Riemannian metric.

Exercise 0.1. Standard spherical coordinates on $M^2 = S^2$, the unit two-sphere, are $\overrightarrow{x}(\theta,\phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$. Compute that $d\overrightarrow{x} \cdot d\overrightarrow{x} = d\phi^2 + \sin^2(\phi)\, d\theta^2$.

$$d\overrightarrow{x} = (-\sin(\theta)\sin(\phi)d\theta + \cos(\theta)\cos(\phi)d\phi,\ \cos(\theta)\sin(\phi)d\theta + \sin(\theta)\cos(\phi)d\phi,\ -\sin(\phi)d\phi).$$

Thus,

$$\begin{aligned} d\overrightarrow{x} \cdot d\overrightarrow{x} &= \sin^2(\theta)\sin^2(\phi)d\theta^2 + \cos^2(\theta)\cos^2(\phi)d\phi^2 - 2\sin(\theta)\sin(\phi)\cos(\theta)\cos(\phi)d\theta d\phi \\ &+ \cos^2(\theta)\sin^2(\phi)d\theta^2 + \sin^2(\theta)\cos^2(\phi)d\phi^2 + 2\sin(\theta)\sin(\phi)\cos(\theta)\cos(\phi)d\theta d\phi \\ &+ \sin^2(\phi)d\phi. \\ &= d\phi^2 + \sin^2(\phi)\, d\theta^2. \end{aligned}$$

Exercise 1.1. Verify that if we apply GS to the coordinate basis $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ associated to Gauss' form

$$ds^2 = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2$$

then we get a smooth frame e_1, e_2. (Express e_1, e_2 in terms of E, F, G and $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$. Argue the resulting vector fields are smooth.)

Note that ds^2 is a positive-definite quadratic form on the tangent bundle. Hence, $EG - F^2 > 0$ everywhere. Let

$$\begin{aligned} v_1 &= \partial_u \\ v_2 &= \partial_v - \frac{\partial_v \cdot \partial_u}{\partial_u \cdot \partial_u}\, \partial_u = \partial_v - \frac{F}{E}\, \partial_u. \end{aligned}$$

Then v_1 and v_2 are orthogonal. We can see that

$$\begin{aligned}
< v_1, v_1 > &= E \\
< v_2, v_2 > &= \frac{F^2}{E} - 2\frac{F^2}{E} + G = \frac{-F^2 + EG}{E}.
\end{aligned}$$

Let

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{E}}\,\partial_u \\
e_2 &= \sqrt{\frac{E}{EG - F^2}}\left(\partial_v - \frac{F}{E}\,\partial_u\right).
\end{aligned}$$

Then e_1 and e_2 are orthonormal. Since $ds^2(\partial_u, \partial_u) = E \neq 0$, e_1 is a smooth vector field. Since $EG - F^2 > 0$ everywhere, $\sqrt{\frac{E}{EG-F^2}}$ is a smooth function. Hence, e_2 is also a smooth vector field.

Exercise 1.2. Let E_1, E_2 be a frame field and θ^1, θ^2 the dual coframe field. Show that E_1, E_2 is orthonormal if and only if $ds^2 = (\theta^1)^2 + (\theta^2)^2$.

($\Rightarrow$) Note that $ds^2 = e(\theta^1)^2 + 2f\theta^1\theta^2 + g(\theta^2)^2$. On the tangent space T_pM for each $p \in M$, the metric tensor $ds^2 : T_pM \times T_pM \to \mathbb{R}$ is a quadratic symmetric bilinear form. For $v, w \in T_pM$, let $v = (v_1, v_2)$ and $w = (w_1, w_2)$ with respect to the basis $E_1(p), E_2(p)$. Then there is a symmetric matrix $\begin{bmatrix} e & f \\ f & g \end{bmatrix} \in GL(2, \mathbb{R})$ such that

$$ds^2_p(v, w) = [v_1 \;\; v_2] \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Then

$$\begin{aligned}
ds^2_p(v, w) &= ev_1w_1 + f(v_1w_2 + v_2w_1) + gv_2w_2 \\
&= [e(\theta^1_p)^2 + 2f\theta^1_p\theta^2_p + g(\theta^2_p)^2](v, w).
\end{aligned}$$

Thus,

$$\begin{aligned}
1 &= ds^2_p(E_1(p), E_1(p)) = e\theta^1_p(E_1(p))\theta^1_p(E_1(p)) = e \\
0 &= ds^2_p(E_1(p), E_2(p)) = 2f\frac{1}{2}[\theta^1(E_1(p))\theta^2(E_2(p)) + \theta^1(E_2(p))\theta^2(E_1(p))] = f \\
1 &= ds^2_p(E_2(p), E_2(p)) = g\theta^2_p(E_1(p))\theta^2_p(E_1(p)) = g.
\end{aligned}$$

Thus, $ds_p^2 = (\theta_p^1)^2 + (\theta_p^2)^2$ on $T_pM.q$

($\Leftarrow$) At each point $p \in M$

$$\begin{aligned} ds_p^2(E_i(p), E_j(p)) &= \theta_p^1(E_i(p))\theta_p^1(E_j(p)) + \theta_p^2(E_i(p))\theta_p^2(E_j(p)) \\ &= \delta_{ij}. \end{aligned}$$

Thus, E_1, E_2 is orthonormal.

Exercise 1.3. Let $M \subset \mathbb{R}^3$ be an oriented surface with unit normal vector N. Show that the two-form $\Omega = i_N(dx \wedge dy \wedge dz)$ restricted to M is the area form dA on M. Hint: show that if $\overrightarrow{v}, \overrightarrow{w} \in T_mM$ then $\Omega(\overrightarrow{v}, \overrightarrow{w}) = N \cdot (\overrightarrow{v} \times \overrightarrow{w})$.

Let $N = (N_x, N_y, N_z)$.

$$\begin{aligned} \Omega = i_N(dx \wedge dy \wedge dz) &= i_N(dx)(dy \wedge dz) - dx \wedge i_N(dy \wedge dz) \\ &= dx(N)(dy \wedge dz) - dx \wedge (dy(N)dz - dz(N)dy) \\ &= N_x(dy \wedge dz) - N_y(dx \wedge dz) + N_z(dx \wedge dy). \end{aligned}$$

Let $\overrightarrow{v} = (v_x, v_y, v_z)$ and $\overrightarrow{w} = (w_x, w_y, w_z)$ in T_mM. Then

$$\begin{aligned} \Omega(\overrightarrow{v}, \overrightarrow{w}) &= N_x(dy \wedge dz)(\overrightarrow{v}, \overrightarrow{w}) - N_y(dx \wedge dz)(\overrightarrow{v}, \overrightarrow{w}) + N_z(dx \wedge dy)(\overrightarrow{v}, \overrightarrow{w}) \\ &= N_x(v_yw_z - w_yv_z) - N_y(v_xw_z - w_xv_z) + N_z(v_xw_y - v_yw_x) \\ &= (N_x, N_y, N_z) \cdot (v_yw_z - w_yv_z, -(v_xw_z - w_xv_z), v_xw_y - v_yw_x) \\ &= N \cdot (\overrightarrow{v} \times \overrightarrow{w}). \end{aligned}$$

For any distinct vectors $\overrightarrow{v}, \overrightarrow{w} \in T_mM$, the vector $\overrightarrow{v} \times \overrightarrow{w} = cN$ for a nonzero constant $c \in \mathbb{R}$. Thus, Ω is a nowhere-vanishing 2-form. Hence, Ω is a volume form on M.

Exercise 1.4. Suppose that θ^1, θ^2 is an oriented orthonormal coframe for M^2, ds^2 defined on a neighborhood $V \subset M$. Show that the pair of one-forms $\overline{\theta^1}, \overline{\theta^2}$, also defined on V, is also an oriented orthonormal coframe if and only if there is a circle-valued function $\psi : V \to S^1$ such that on V we have that:

$$\begin{aligned} \overline{\theta^1} &= \cos(\psi)\theta^1 + \sin(\psi)\theta^2 \\ \overline{\theta^2} &= -\sin(\psi)\theta^1 + \cos(\psi)\theta^2 \end{aligned}$$

($\Rightarrow$) Let

$$\overline{\theta^1} = f_1\theta^1 + f_2\theta^2$$
$$\overline{\theta^2} = g_1\theta^1 + g_2\theta^2$$

Since $ds^2 = (\overline{\theta^1})^2 + (\overline{\theta^2})^2$ and $ds^2 = (\theta^1)^2 + (\theta^2)^2$,

$$(f_1)^2 + (g_1)^2 = 1 \quad , \ (f_2)^2 + (g_2)^2 = 1 \quad \text{and} \ \ f_1 f_2 + g_1 g_2 = 0.$$

Also, $f_1 g_2 - g_1 f_2 = 1$ because $\theta_1 \wedge \theta_2 = \overline{\theta_1} \wedge \overline{\theta_2}$. Thus, for each $p \in M$ and on a neighborhood V of p, the matrix

$$\begin{bmatrix} f_1(p) & g_1(p) \\ f_2(p) & g_2(p) \end{bmatrix} \in SO(2, \mathbb{R}).$$

Then there is a circle-valued function $\psi : V \to S^1$ such that

$$\begin{bmatrix} f_1(p) & g_1(p) \\ f_2(p) & g_2(p) \end{bmatrix} = \begin{bmatrix} \cos(\psi(p)) & -\sin(\psi(p)) \\ \sin(\psi(p)) & \cos(\psi(p)) \end{bmatrix}.$$

Thus, on V

$$\overline{\theta^1} = \cos(\psi)\theta^1 + \sin(\psi)\theta^2$$
$$\overline{\theta^2} = -\sin(\psi)\theta^1 + \cos(\psi)\theta^2.$$

($\Leftarrow$) We can easily see that $(\overline{\theta^1})^2 + (\overline{\theta^2})^2 = (\theta^1)^2 + (\theta^2)^2 = ds^2$ and $\overline{\theta^1} \wedge \overline{\theta^2} = \theta^1 \wedge \theta^2$.

Exercise 2.1. Given the oriented orthonormal coframe θ^1, θ^2, show that the Cartan's structure formula uniquely determines ω.

Let $\omega = \alpha\theta^1 + \beta\theta^2$ for scalar function α, β. Then

$$d\theta^1 = \omega \wedge \theta^2 = (\alpha\theta^1 + \beta\theta^2) \wedge \theta^2 = \alpha \ \theta^1 \wedge \theta^2$$
$$d\theta^2 = -\omega \wedge \theta^2 = -(\alpha\theta^1 + \beta\theta^2) \wedge \theta^1 = \beta \ \theta^1 \wedge \theta^2.$$

Let $\omega' = \alpha'\theta^1 + \beta'\theta^2$ be another one-form satisfying the Cartan's structure equation. Then $\alpha' = \alpha$ and $\beta' = \beta$.

Exercise 2.2. Show that if $\overline{\theta^1}, \overline{\theta^2}$ is another oriented orthonormal coframe then its connection one-form $\overline{\omega}$ is given by $\overline{\omega} = \omega + d\psi$ with ψ as in exercise 1.4.

By Exercise 1.4, there is a circle-valued function ψ such that

$$\begin{aligned}\overline{\theta^1} &= \cos(\psi)\theta^1 + \sin(\psi)\theta^2 \\ \overline{\theta^2} &= -\sin(\psi)\theta^1 + \cos(\psi)\theta^2.\end{aligned}$$

Then

$$\begin{aligned}d\overline{\theta^1} &= -\sin(\psi)d\psi \wedge \theta^1 + \cos(\psi)d\theta^1 + \cos(\psi)d\psi \wedge \theta^2 + \sin(\psi)d\theta^2 \\ &= -\sin(\psi)(d\psi + \omega) \wedge \theta^1 + \cos(\psi)(d\psi + \omega) \wedge \theta^2 \\ &= (d\psi + \omega) \wedge (-\sin(\psi)\theta^1 + \cos(\psi)\theta^2) \\ &= (d\psi + \omega) \wedge \overline{\theta^2} \\ d\overline{\theta^2} &= -\cos(\psi)d\psi \wedge \theta^1 - \sin(\psi)d\theta^1 - \sin(\psi)d\psi \wedge \theta^2 + \cos(\psi)d\theta^2 \\ &= -\cos(\psi)(d\psi + \omega) \wedge \theta^1 - \sin(\psi)(d\psi + \omega) \wedge \theta^2 \\ &= -(d\psi + \omega) \wedge (\cos(\psi)\theta^1 + \sin(\psi)\theta^2) \\ &= (d\psi + \omega) \wedge \overline{\theta^1}.\end{aligned}$$

Thus, the connection one-form $\overline{\omega}$ is given by $\overline{\omega} = \omega + d\psi$.

Exercise 2.3. Show that K does not depend on the orientation. If we reverse the orientation of M then K remains unchanged. In particular M does not need to be oriented for the Gaussian curvature to be defined.

Assume that M is oriented. Then M has an oriented orthonormal coframe (θ^1, θ^2) on a connected neighborhood U of p, for $p \in M$.The 2-form $\theta^1 \wedge \theta^2$ determines the orientation of M. Also, $-\theta^1 \wedge \theta^2$ determines the reversing orientation of M. Let $\overline{\theta^1} = -\theta^1$ and $\overline{\theta^2} = \theta^2$. Then

$$ds^2 = (\overline{\theta^1})^2 + (\overline{\theta^2})^2.$$

Also,

$$\begin{aligned}d\overline{\theta^1} &= -\omega \wedge \overline{\theta^2} \\ d\overline{\theta^2} &= -\omega \wedge \theta^2 = \omega \wedge \overline{\theta^1}.\end{aligned}$$

Let $\overline{\omega} = -\omega$. Then $d\overline{\omega} = -K\overline{\theta^1} \wedge \overline{\theta^2}$. Thus, K remains unchanged.

Exercise 2.4. Compute the Cartan structure equations and find the Gaussian curvature K in the following two cases :

$$ds^2 = dr^2 + f(r)^2 d\theta^2 \tag{1}$$
$$ds^2 = \lambda(u,v)^2(du^2 + dv^2). \tag{2}$$

(1) Assume that $f(r)$ is a nowhere vanishing function. Let $\theta_1 = dr$ and $\theta_2 = f(r)d\theta$.

$$d\theta_1 = 0$$
$$d\theta_2 = -\left(\frac{\partial f}{\partial r} d\theta\right) \wedge dr.$$

Let $\omega = \frac{\partial f}{\partial r} d\theta$. Then $d\omega = \frac{\partial^2 f}{\partial r^2} dr \wedge d\theta = \frac{\partial^2 f}{\partial r^2} \cdot \frac{1}{f(r)} \theta_1 \wedge \theta_2$. Thus, the curvature is $-\frac{\partial^2 f}{\partial r^2} \cdot \frac{1}{f(r)}$.

(2) Assume that $\lambda(u,v)$ is a nowhere vanishing function. Let $\theta_1 = \lambda(u,v)du$ and $\theta_2 = \lambda(u,v)dv$.

$$d\theta_1 = -\frac{\partial \lambda}{\partial v} du \wedge dv = \left(-\frac{\partial \lambda}{\partial v} \cdot \frac{1}{\lambda} du\right) \wedge \lambda dv = \left(-\frac{\partial \lambda}{\partial v} \cdot \frac{1}{\lambda} du + \frac{\partial \lambda}{\partial u} \cdot \frac{1}{\lambda} dv\right) \wedge \theta_2$$
$$d\theta_2 = \frac{\partial \lambda}{\partial u} du \wedge dv = \left(-\frac{\partial \lambda}{\partial u} \cdot \frac{1}{\lambda} dv\right) \wedge \lambda du = -\left(-\frac{\partial \lambda}{\partial v} \cdot \frac{1}{\lambda} du + \frac{\partial \lambda}{\partial u} \cdot \frac{1}{\lambda} dv\right) \wedge \theta_1.$$

Let $\omega = -\frac{\partial \lambda}{\partial v} \cdot \frac{1}{\lambda} du + \frac{\partial \lambda}{\partial u} \cdot \frac{1}{\lambda} dv = (\log \lambda)_u \, dv - (\log \lambda)_v \, du$. Then

$$d\omega = [(\log \lambda)_{uu} + (\log \lambda)_{vv}] \, du \wedge dv = \triangle(\log \lambda) \cdot \frac{1}{\lambda^2} \theta_1 \wedge \theta_2.$$

Thus,

$$K = - \triangle (\log \lambda) \cdot \frac{1}{\lambda^2}.$$

Exercise 2.6. Find f as in (1) of exer 2.4 such that $K = -1$ and f has first order Taylor expansion $f(r) = r + \mathcal{O}(r^2)$.

From (1) in exercise 2.4, $\frac{\partial^2 f}{\partial r^2} \cdot \frac{1}{f(r)} = 1$. Then $\frac{\partial^2 f}{\partial r^2} = f(r)$. Hence,

$$f(r) = c_1 e^r + c_2 e^{-r}$$

for any constants c_1, c_2. By the assumption, $f(0) = 0$ and $f'(0) = 1$.

$$0 = f(0) = c_1 + c_2$$
$$1 = f'(0) = c_1 - c_2.$$

Hence, $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Thus, $f(r) = \sinh(r)$.