Math 209 Homework 3

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Prove that if $\alpha_1, \dots, \alpha_n$ are elements of V^*, where $\alpha_i = \sum_{j=1}^{n} a_{ij} e_j^*$, then $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n = det[a_{ij}] e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*$.

By multilinearity of the wedge operation,

$$\alpha_1 \wedge \dots \wedge \alpha_n = (\sum_{j=1}^{n} a_{1j} e_j^*) \wedge (\sum_{j=1}^{n} a_{2j} e_j^*) \wedge \dots \wedge (\sum_{j=1}^{n} a_{nj} e_j^*) = \sum_{I} a_{1i_1} a_{2i_2} \dots a_{ni_n} e_{i_1}^* \wedge e_{i_2}^* \wedge \dots \wedge e_{i_n}^*$$

where $I = (i_1, \dots, i_n)$ varies over all possible ordered multi-indices, $1 \leq i_j \leq n$. Since $e_i^* \wedge e_i^* = 0$, we only need to consider multi-indices which have no repeats. All such multi-indices are given by permutations of $(1, 2, \dots, n)$, and every permutation $\sigma \in S_n$ gives a multi-index $I_\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$, so we can rewrite the above equation as:

$$\alpha_1 \wedge \dots \wedge \alpha_n = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} e_{\sigma(1)}^* \wedge e_{\sigma(2)}^* \wedge \dots \wedge e_{\sigma(n)}^*$$

Since the wedge operation is alternating, $e_{\sigma(1)}^* \wedge e_{\sigma(2)}^* \wedge \dots \wedge e_{\sigma(n)}^* = (-1)^{sgn(\sigma)} e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*$, so we get

$$\alpha_1 \wedge \dots \wedge \alpha_n = (\sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}) e_1^* \wedge e_2^* \wedge \dots \wedge e_n^* = det([a_{ij}]) e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*$$

Where the last equality comes directly from the definition of the determinant of a matrix.