Homework 7 - Math 209

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Exercise 1.

Proof. Let θ be a non-vanishing form such that $\theta \wedge d\theta = 0$ everywhere. Then, we know that $d\theta = \theta \wedge \alpha$ for some one-form α. This is because of the following (let X be a vector):

$$\iota_X(\theta \wedge d\theta) = (\iota_X\theta) \wedge d\theta - \theta \wedge (\iota_X d\theta) = \theta(X) \wedge d\theta - \theta \wedge (\iota_X d\theta)$$

where $\iota_X(\theta) = \theta(X)$. Now, $\iota_X(\theta \wedge d\theta) = \iota_X(0) = 0$ and so we choose X to be dual to θ and we get

$$0 = d\theta - \theta \wedge \alpha$$

where $\alpha = \iota_X(d\theta)$ and so $d\theta = \theta \wedge \alpha$.

We note that $\ker(\theta)$ defines a smooth distribution by the 1-form Criterion for Smooth Distributions (this means that if we have an n-dimensional manifold, then we have a rank k distribution D if and only if there are $n-k$ one-forms such that $D_q = \bigcap_{i=1}^{n-k} \ker \omega^i \mid_q$). We further note that

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]) = X(0) - Y(0) - \theta([X,Y]) = -\theta([X,Y]).$$

We also have $d\theta(X,Y) = \theta \wedge \alpha(X,Y) = \theta(X)\alpha(Y) - \theta(Y)\alpha(X) = 0 - 0 = 0$. Thus, we have $\theta([X,Y]) = 0$ and so our distribution is involutive and therefore completely integrable by the Frobenius Theorem.

Since D is completely integrable, by definition, for all $q \in \mathbb{R}^3$, there exists a smooth coordinate chart (U, φ) such that $\varphi(U)$ is a cube in $\mathbb{R}^3$ and for all $q \in U$, D_q is spanned by $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. So, $T_q\mathbb{R}^3$ is spanned by $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)$, where z is given by $\varphi(q) = (x(q), y(q), z(q))$. Now, dz_q has the same kernel as θ_q. Note that since they are both linear functionals on $T_q\mathbb{R}^3$, they are multiples of each other.

Let $\{U_q, \varphi_q\}$ be the flat charts that cover $\mathbb{R}^3$ and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_q\}$. On each coordinate chart, let $g_q = z_q$, so $dg_q = dz_q$. Further, let $\omega = \sum \rho_\alpha dg_q$. We have

$$d(\sum \rho_\alpha dg_q) = \sum (d\rho_\alpha) \wedge dz_q = d(\sum \rho_\alpha) \wedge dz_q = d(1) \wedge dz_q = 0.$$

We have linearity, because at each point q, the sum is a finite sum. So, we see that ω is a closed form and as $H^1_{\mathrm{dR}}(\mathbb{R}^3)$ is trivial, that means ω is an exact form, therefore, there exists a g such that $\omega = dg$.

Now, we have to show that ω and θ have the same kernel, they they are multiples of each other (again, because at a point they are linear functionals) and this will mean that $\theta = f dg$.

Now, it's clear that $\ker \theta \subset \ker \omega$, by construction of ω.

So, we must show that $\ker \omega \subset \ker \theta$. So, to this end, we want to show if we have two charts, (U_p, φ_p) and (U_q, φ_q) containing points p and q, respectively and such that $U_p \cap U_q \neq \varnothing$, then $dz_q = \lambda dz_p$, where $\lambda \neq 0$ is positive. But we can do this, because of the orientability of $\mathbb{R}^3$. Therefore, there exists, $f, g \in C^\infty(\mathbb{R}^3)$ such that $\theta = f dg$. □

Exercise 2.

Proof. The proof outlined above will not work if the manifold is not orientable or there exists a closed one-form that is not exact, i.e. the first de Rham cohomology group is not trivial. □

Exercise 3.

Proof. We did not use the dimension in the argument for Exercise 1, so the proof will carry over to $\mathbb{R}^n$, because it is orientable and $H^1_{\mathrm{dR}}(\mathbb{R}^n) = 0$. □

Exercise 4.

Proof. This will not work for the same reasons we outlined in Exercise 2. □