# MATH 209, MANIFOLDS II, WINTER 2014

## Solutions.

Below, $F, G$ are vector fields, $\alpha, \beta$ are forms and $f, g$ is a (scalar) function. $\nabla$ is the differential operator of vector calculus so that $\nabla f = grad(f)$, $\nabla \times F = curl(F)$ and $\nabla \cdot F = div(F)$.

TABLE 1. Comparison of vector calculus and exterior differential calculus

| Vector Calculus Formulae | Exterior Differential Calculus |
|---|---|
| $\mathbb{R}^3$ | manifold$M^n$ |
| linearity for $\nabla$ : | linearity for $d$ : |
| $\nabla(f + cg) = \nabla f + c\nabla g$ | $d(\alpha + c\beta) = d\alpha + cd\beta$ |
| $\nabla \times (F + cG) = \nabla \times F + c\nabla \times G$ | same as above |
| $\nabla \cdot (F + cG) = \nabla \cdot G + c\nabla \cdot G$ | same as above |
| Liebnitz for $\nabla$ | Liebnitz for $d$ |
| $\nabla(fg) = f\nabla g + g\nabla g$ | $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\lvert\alpha\rvert}\alpha \wedge d\beta$ |
| $\nabla \times (fF) = f\nabla \times F + \times \nabla f \times F$ | same as above |
| $\nabla \cdot (fF) = \nabla f \cdot F + \nabla f \cdot F$ | same as above |
| mixed partials commute | $d^2 = 0$ : |
| $\nabla \times (\nabla f) = 0$ | $d^2\alpha = 0$ |
| $\nabla \cdot (\nabla \times F) = 0$ | same as above |
| $- - - - - - - - - - - - - - - - - - -$ | Poincairé lemma, which says: |
| If the domain of $F$ is $\mathbb{R}^3$ or a contractible open subset thereof: | If the domain of $\alpha$ is $\mathbb{R}^n$ or a contractible open subse |
| curl-free implies conservative: $\nabla \times F = 0f \implies F = \nabla f$ | $d\alpha = 0 \implies \alpha = d\beta$ |
| div.-free implies a curl: $\nabla \cdot F = 0 \implies F = \nabla \times G$ | same |
| and the integral identities! | Stokes' formula |
| Stokes : $\int_S (\nabla \times F) \cdot ndS = \int_C F \cdot ds$ | $\int_\Sigma d\alpha = \int \partial\Sigma\alpha$ |
| Divergence or Gauss Thm: $\int_R (\nabla \cdot F)dV = \int_S F \cdot ndS$ | same as above |
| Laplacian:$\Delta = \nabla^2 = div(grad)$ | HODGE Laplacian. Requires a Riem metric $g$ |
| $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ | $\Delta = (d + \delta)^2; \delta = d^* = (-1)^X * d*$ |
| $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$ | $\Delta(fg) = f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f$ |