HW 2 solution commentary.

Exer 1. Better than using "the extension lemma 2.2.6" is to simply use a bump function. Why better? Well think of the prelim. No one outside of someone currently reading Lee knows this extension lemma by heart. But everyone grading a prelim in topology - geometry knows bump functions.

To make the local section nonzero at p and vanishing outside a (small) neighborhood of p choose a local trivializing nbhd U of p and a small open $p \in V \subset U$ with V not containing q. Choose a bump function β which is 1 at p and zero off of V. Over U we have a local trivializaiton $E_U \cong U \times \mathbb{R}^k$. Pick any NONZERO vector $e \in \mathbb{R}^k$. For example $e = e_1 = (1, 0, 0, \ldots, 0)$. Then $\phi^{-1}(u, e) := s(u)$ is nonvanishing on E_U. (WHY???) And βs makes global sense as a section of all of $E \to M$ since it is dead zero off of U.

Exer 2. The hard direction here is to show that if $E_{\mathbb{R}}$ is trivial then so it $E_{\mathbb{C}}$.

A key here is to understanding what it means for two bundles "to be the same" and to realize that this concept is different depending on whether the bundle is viewed as a real vector bundle or a complex vector bundle. An ($\mathbb{K}$) bundle ($\mathbb{K}$ a field, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ here) $E \to M$ is trivial if and only if there is a $\mathbb{K}$-vector bundle isomorpism $\phi : E \cong M \times \mathbb{V}$ where $\mathbb{V}$ is a vector space over $\mathbb{K}$. This means that ϕ is a diffeomorphism, commutes with the projection, and most importantly here, is *is* $\mathbb{K}$ *linear* on the fibers. A $\mathbb{C}$-linear map $\mathbb{V}_p \cong \mathbb{C}$ is automatically $\mathbb{R}$ linear, but $\mathbb{R}$-linear maps $\mathbb{V}_p$ to $\mathbb{R}^2$ (or $\mathbb{C} \to \mathbb{R}^2$) need not be $\mathbb{C}$-linear and this lack is the heart of the subtlety of this problem.

As topological spaces with projections, $E_{\mathbb{R}}$ and $E_{\mathbb{C}}$ are the same space. To say $E_{\mathbb{C}}$ is trivial then means we have a global trivialization $E \cong M \times \mathbb{C}$ which is $\mathbb{C}$-linear on each fiber. To say that $E_{\mathbb{R}}$ is trivial means that we have a global trivialization $E \cong M \times \mathbb{R}^2$ which is an $\mathbb{R}$-linear isomorphism on each fiber.

To get from $E_{\mathbb{C}}$ to $E_{\mathbb{R}}$ use the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$. This is a real linear identification.

To get from $E_{\mathbb{R}}$ to $E_{\mathbb{C}}$ is trickier because somehow we have to "twist" the real linear automorphism and make it complex linear.

At this stage a BASIC FACT about line bundles over $\mathbb{K}$ comes in to play. A line bundle is trivial if and only if it admits a non-vanishing section. [PROVE THIS BASIC FACT!]

Now the exer finishes off simply. Since $E_{\mathbb{R}}$ is globally trivial it admits a non-vanishing section. (WHICH? HOW?) Use this section as per BASIC FACT to build the global *complex linear* trivialization of $E_{\mathbb{C}}$

Related facts: A line bundle is locally trivial over an open set U is trivial if and only if it admits a non-vanishing section over U. A rank k vector bundle is trivial if and only if it admits k *linearly independent sections.* A rank k vector bundle is trivial over U if and only if it admits k *linearly independent sections* defined over U.