BOB HINGTGEN

Bob H.

2). Let α be a one-form on a smooth-connected manifold M. Recall that α is said to be exact if there exists a function f such that $\alpha = df$. Prove that

$$\alpha \text{ is exact} \iff \int_\gamma \alpha = 0 \quad \text{for any closed curve } \gamma.$$

Pf: $\Longrightarrow$ Assume that α is exact, i.e. there exists f such that $\alpha = df$ and let $\gamma : [a,b] \to M$ be a smooth curve.

Let $(\gamma^1(t), \ldots, \gamma^n(t))$ be a coordinate representation of γ.

$$(\gamma^* df)_t = \gamma^* df_{\gamma(t)} = \gamma^* \left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} \, dx_i \right)_{\gamma(t)}$$

$$= \gamma^* \left(\sum_{i=1}^n \frac{\partial f}{\partial x^i}(\gamma(t)) \, dx_i \big|_{\gamma(t)} \right)$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(\gamma(t)) \, \gamma^* dx_i \big|_{\gamma(t)} = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(\gamma(t)) \, (\gamma^* dx_i)_t$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(\gamma(t)) \, (d(x_i \circ \gamma))_t = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(\gamma(t)) \, (d\gamma^i)_t$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(\gamma(t)) \, \frac{d\gamma^i}{dt}(t) \, dt = \frac{d}{dt}(f \circ \gamma) \Big|_t$$

well done!

Thus, $$\int_\gamma \alpha = \int_\gamma df = \int_{[a,b]} \gamma^* df = \int_a^b \frac{d}{dt}(f \circ \gamma) \, dt$$

which implies $$\int_\gamma \alpha = f \circ \gamma(b) - f \circ \gamma(a).$$ Perfect

Thus if γ is closed ($\gamma(a) = \gamma(b)$), then $\int_\gamma \alpha = 0$.

1. Let M be a smooth manifold, and $f \in C^\infty(M)$. Then by the natural coordinate on $\mathbb{R}$, we mean the identity map $\mathbb{R} \to \mathbb{R}$ by $y \mapsto y$. For $v \in \mathfrak{X}(M), p \in M$, we define the one-form df by $df(v_p) := v_p(f)$ where $v_p \in T_pM$ is thought of as a derivation.

   Now we have $f^*dy(v_p) = dy(f_*(v_p)) = f_*v_p(y) = v_p(y \circ f) = v_p(f) = df(v_p)$, so $f^*dy = df$ since v and p were arbitrary.

   since $y \circ f = f$.

   Let $(U, (x_1, ..., x_n))$ be a system of local coordinates on M. For $p \in U$, define the one-forms dx_i as the dual basis to the basis $\frac{\partial}{\partial x_i}|_p$ of T_pM.

   Since dx_i is a basis for T_p^*M, we may write $df(v_p) = \sum_1^n a_i dx_i(v_p)$ for some $a_i \in \mathbb{R}$ and any $v \in \mathfrak{X}(M)$. Now $df(\frac{\partial}{\partial x_k}|_p) = \sum_{i=1}^n a_i dx_i(\frac{\partial}{\partial x_k}|_p) = a_k$ by definition of dual basis. On the other hand, by our above definition, $df(\frac{\partial}{\partial x_k}|_p) = \frac{\partial}{\partial x_k}|_p(f) = \frac{\partial f}{\partial x_k}|_p$, so $a_k = \frac{\partial f}{\partial x_k}|_p$, hence $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$ on U.

2. Let α be a one form on a smooth connected manifold M.
   If α is exact, then $\alpha = df$ for some $f \in C^\infty(M)$. Let γ be a closed curve, by Stoke's theorem: pf? see Bob's.

   $\int_\gamma \alpha = \int_\gamma df = \int_{\partial\gamma} f = \int_\emptyset f = 0$ (since the boundary of a closed curve is empty).

   well done!

   Conversely suppose $\int_\gamma \alpha = 0$ for any closed curve γ. Fix any $x_0 \in M$, for $x \in M$ let c be a curve from x_0 to x (since M is connected and connected implies path connected for manifolds our curve c exists) define $f(x) = \int_c \alpha$. Then f is well-defined, for if $c_1, c_2 : [0,1] \to M$ are two curves from x_0 to x, then set $\gamma(t) = \begin{cases} c_1(t), & t \in [0,1] \\ c_2(2-t) & t \in [1,2] \end{cases}$, so γ is a closed curve.

   By assumption $0 = \int_\gamma \alpha = \int_0^2 \alpha(\gamma'(t))dt = \int_0^1 \alpha(c_1'(t))dt + \int_1^2 \alpha(c_2'(2-t))dt = \int_{c_1} \alpha - \int_0^1 \alpha(c_2'(s))ds = \int_{c_1} \alpha - \int_{c_2} \alpha \Rightarrow \int_{c_1} \alpha = \int_{c_2} \alpha$ and f is well-defined.

   We claim $df = \alpha$. First we'll do it for $M = \mathbb{R}^n$, then we'll get the general result via a chart.

   For $M = \mathbb{R}^n$ and $y \in \mathbb{R}^n$, let η be a path from x_0 to y and define $c_t : [0,t] \to \mathbb{R}^n$ by $c_t(s) = y + sx_i$ (where x_i is our standard basis for $\mathbb{R}^n$) write $\alpha = \sum_{i=1}^n a_i dx_i$ where $a_i \in C^\infty(\mathbb{R}^n)$. Then:

$$\frac{\partial f}{\partial x_i}\bigg|_y = \frac{d}{dt}\bigg|_{t=0} f(y + tx_i) = \frac{d}{dt}\bigg|_{t=0} \int_{c_t} \alpha + \int_\eta \alpha = \frac{d}{dt}\bigg|_{t=0} \int_0^t \alpha(c_t'(s))\ ds =$$

$$\frac{d}{dt}\bigg|_{t=0} \int_0^t \alpha(x_i)\ ds = \frac{d}{dt}\bigg|_{t=0} \int_0^t a_i(y + sx_i)ds = a_i(y + tx_i)\bigg|_{t=0} = a_i(y).$$

   So $df = \alpha$ by # 1.

   Now for a general manifold, let $p \in M$, and $(U, \varphi : \mathbb{R}^n \to U)$ be a chart with $p \in U$, set $y = \varphi^{-1}(p)$. We have seen that using a different base point for the definition of f changes f by a constant which has no effect on df, so we may chose our base point

   where?

$x_0 \in U$. Then for a curve γ from x_0 to p, $\varphi^* f(y) = f\varphi(y) = f(p) = \int_\gamma \alpha = \int_{\varphi(\gamma)} \varphi^*\alpha$, and $\varphi(\gamma)$ is a curve from $\varphi^{-1}(x_0)$ to y so by our $\mathbb{R}^n$ case, $d\varphi^* f = \varphi^*\alpha$. Now:

$\varphi^*\alpha = d\varphi^* f = \varphi^* df \Rightarrow (\varphi^{-1})^*\varphi^*\alpha = (\varphi^{-1})^*\varphi^* df \Rightarrow (\varphi^{-1}\varphi)^*\alpha = (\varphi^{-1}\varphi)^* df \Rightarrow \alpha = df$.

So we have that α is exact $\iff$ $\int_\gamma \alpha = 0$ for any closed curve γ.

3. Let $\alpha = xdy - ydx$ on $\mathbb{R}^2$. Let S^1_R be circle of radius $R > 0$ centered at origin and oriented widdershins. Then $c(t) = (R\cos t, R\sin t)$ for $t \in [0, 2\pi]$ is a parametrization of S^1_R with $c'(t) = (-R\sin t, R\cos t)$.

   So $\int_{S^1_R} \alpha = \int_0^{2\pi} \alpha(c'(t))\ dt = \int_0^{2\pi} R\cos t dy(c'(t)) - R\sin t dx(c'(t))\ dt =$ $\int_0^{2\pi} R^2\cos^2 t + R^2 \sin^2 t\ dt = 2\pi R^2 \neq 0$, hence α is not exact by # 2.

   For any closed simple curve γ in $\mathbb{R}^2$, we have that γ is the boundary of some region that is $\gamma = \partial U$. Stoke's theorem gives: $\int_\gamma \alpha = \int_{\partial U} \alpha = \int_U d\alpha = \int_U dx \wedge dy - dy \wedge dx =$ $2\int_U dx \wedge dy = 2\text{Area}(U)$, so integrating α over γ gives twice the area of the region enclosed by γ.

Danquynh

This shows that we can restrict our consideration to a local chart U. In other words, we may proceed using $\tilde{f}$ instead of f.

With $\gamma(t) = (0, \dots, t, \dots, 0)$, $\gamma'(t) = \frac{\partial}{\partial x^j}\Big|_{\gamma(t)}$.

Also, $\alpha_{\gamma(t)} = \sum \alpha_i(\gamma(t))\, dx^i$. Thus,

$$\alpha_{\gamma(t)}(\gamma'(t)) = \sum \alpha_i(\gamma(t))\, dx^i\left(\frac{\partial}{\partial x^j}\Big|_{\gamma(t)}\right)$$

$$= \alpha_j(\gamma(t)) \qquad (2.1)$$

(since $dx^i\left(\frac{\partial}{\partial x^j}\Big|_{\gamma(t)}\right) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$)

Since $\gamma|_{[-\varepsilon, t]}$ is a smooth curve from p_1 to $\gamma(t)$ on M (in particular, in U), we have:

$$\tilde{f} \circ \gamma(t) = \int_{p_1}^{\gamma(t)} \alpha = \int_{-\varepsilon}^{t} \alpha_{\gamma(s)}(\gamma'(s))\, ds$$

$$= \int_{-\varepsilon}^{t} \alpha_j(\gamma(s))\, ds \qquad (\text{by } (2.1))$$

$\tilde{f}(\gamma(0)) = ?$

$\tilde{f}(\gamma(-\varepsilon)) = ?$

So, $\frac{\partial \tilde{f}}{\partial x^j}(q_0) = \gamma'(0)\tilde{f} = \frac{d}{dt}\Big|_{t=0} (\tilde{f} \circ \gamma)(t)$

ba'l good!

$$= \frac{d}{dt}\Big|_{t=0} \int_{-\varepsilon}^{t} \alpha_j(\gamma(s))\, ds$$

$$= \left(\alpha_j(\gamma(s))\Big|_t\right)_{t=0} \qquad (\text{by the Fundamental Theorem of Calculus})$$

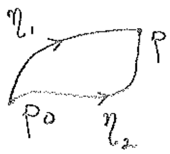
$$= \alpha_j(\gamma(0)) = \alpha_j(q_0)$$

So, $\frac{\partial \tilde{f}}{\partial x^j} = \alpha_j$ and therefore $d\tilde{f} = \alpha$.

As discussed earlier, since $f(q) - \tilde{f}(q) = c \ \forall q \in M$, we have $df = \alpha$. This completes the proof. $\square$

Danquynh

We first show that f is well-defined.



Suppose η_1 and η_2 are curves connecting p and p_0, then $\eta_1 - \eta_2$ is a closed curve.

precise def? How to subtract curves?

By our assumption, $\int_{\eta_1 - \eta_2} \alpha = 0$

$$\int_{\eta_1} \alpha - \int_{\eta_2} \alpha = 0$$

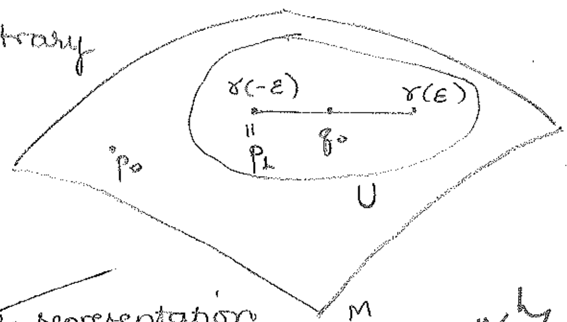
So, $$\int_{\eta_1} \alpha = \int_{\eta_2} \alpha$$

So, $f(p)$ is independent of the curve η chosen.

That is, f is well-defined.

Now let $q_0 \in M$ be arbitrary and choose a smooth chart centered at q_0

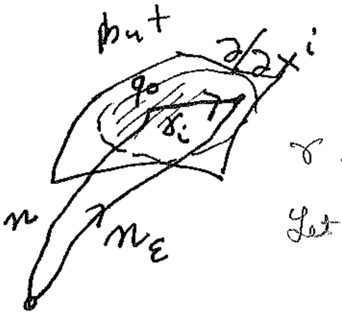
Call it $(U, (x^i))$



why $-\varepsilon$?

In local coordinates, the representation of α in U is: $\alpha = \sum \alpha_i dx^i$

Easier same trick but



Fixing j, define $\gamma : [-\varepsilon, \varepsilon] \to U$ as follows:

$$\gamma(t) = (0, \dots, \underset{\substack{\uparrow \\ \text{the } j^{th} \text{ coordinate}}}{t}, \dots, 0)$$

γ is a smooth curve segment and $Im(\gamma) \subseteq U$.

Let $p_1 = \gamma(-\varepsilon)$. Let $\tilde{f} : M \to \mathbb{R}$ be defined by

$$\tilde{f}(q) = \int_{p_1}^{q} \alpha$$

Then $\forall q \in M$:

$$f(q) - \tilde{f}(q) = \int_{p_0}^{q} \alpha - \int_{p_1}^{q} \alpha = \int_{p_0}^{p_1} \alpha = \text{constant } c$$

Danquynl

Now consider $f = x_j$ for some $j \in \{1, \dots, n\}$, $x_j : U \to \mathbb{R}$:

$$df_p = dx_j\Big|_p = \sum_{i=1}^{n} \frac{\partial x_j}{\partial x_i}(p)\, \lambda^i\Big|_p \qquad (\text{by } (1.2))$$

$$= 0 + \dots + \underbrace{\frac{\partial x_j}{\partial x_j}}_{1}(p)\, \lambda^j\Big|_p + 0 + \dots + 0$$

✓

So, $dx_j = \lambda^j$

Then (1.2) can be rewritten as:

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\, dx_i$$

In general, $\boxed{df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i}$ ✓

□

(2) Let α be a one-form on a smooth connected manifold M.

α is exact $\Leftrightarrow$ $\int_\gamma \alpha = 0$ for any closed curve γ.

Proof:

($\Rightarrow$) Suppose that α is exact, so there exists a function $f \in C^\infty(M)$ such that $\alpha = df$. Let $\gamma : [a,b] \to M$ be a closed curve (i.e $\gamma(a) = \gamma(b)$), then:

$$\int_\gamma \alpha = \int_\gamma df = \int_a^b df_{\gamma(t)}(\gamma'(t))\, dt$$

$$= \int_a^b (f \circ \gamma)'(t)\, dt$$

$$= f(\gamma(b)) - f(\gamma(a)) = 0 \quad (\text{since } \gamma(a) = \gamma(b)).$$

($\Leftarrow$) Suppose that $\int_\gamma \alpha = 0$ for any closed curve γ.

Let $p_0 \in M$ be a reference point and define:

$$f : M \longrightarrow \mathbb{R}$$

$$f(p) = \int_\eta \alpha = \int_{p_0}^{p} \alpha$$

where η is any curve connecting p_0 and p.

Danquynh

$$
\begin{aligned}
(f^* dy)_p(v) &= \left[df_p^* \left(dy_{f(p)} \right) \right](v) \\
&= dy_{f(p)} \left(df_p(v) \right) && \text{(by definition of } df_p^* \text{)} \\
&= df_p(v)\, y && \text{(by definition of } dy \text{)} \\
&= v(y \circ f) && \text{(by definition of } df_p \text{)} \\
&= d(y \circ f)_p (v) && \text{(by definition of } d(y\circ f)\text{)} \\
&= d\, f_p(v)
\end{aligned}
$$

So, $(f^* dy)_p = df_p$ for any $p \in M$

$\therefore$ $\boxed{f^* dy = df.}$

$\square$

b) Let $(x_1, \dots, x_n)$ be a system of local coordinates on M. Then, in local coordinates, $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$

Proof:

For the system of local coordinates $(x_1, \dots, x_n)$ on M, let $(\lambda^1, \dots, \lambda^n)$ be the corresponding local coordinate coframe on M (i.e. each λ^i is a covector field and for each $p \in U \subseteq M$, the n-tuple $(\lambda^1|_p, \dots, \lambda^n|_p)$ forms a basis for $T_p^* M$.)

The representation of df in local coordinates:

$$df_p = \sum_{i=1}^{n} A_i(p)\, \lambda^i|_p \qquad (1.1)$$

where $A_i : U \to \mathbb{R}$ is given by:

$$
\begin{aligned}
A_i(p) &= df_p\left(\frac{\partial}{\partial x_i}\Big|_p \right) = \frac{\partial}{\partial x_i}\Big|_p (f) \quad \text{(by def. of } df\text{)} \\
&= \frac{\partial f}{\partial x_i}\Big|_p = \frac{\partial f}{\partial x_i}(p)
\end{aligned}
$$

So, (1.1) can be written as:

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\, \lambda^i|_p \qquad (1.2)$$

3 very good.

Leo S.
Ryan C.
Shawn T.
Jonathan C.

MATH 209 - MANIFOLDS II, WINTER 2014

HOMEWORK 1 : One-forms & Integration

Lanquynh Nguyen

(1) Let f be a smooth function on a manifold M. Let y be the natural coordinate on $\mathbb{R}$.

a) $df = f^* dy$.

Proof:

We have: $M \xrightarrow{f} \mathbb{R} \xrightarrow{y} \mathbb{R}$. Since y is the natural coordinate on $\mathbb{R}$, y is smooth. For $p \in M$, consider an arbitrary $v \in T_p M$. Recall the definition of the differential of f, which is df:

$$df_p : T_p M \longrightarrow \mathbb{R}$$
$$df_p(v) = vf$$

Let us also recall the definition of the pullback:

$$f : M \longrightarrow \mathbb{R}$$
$$df_p : T_p M \longrightarrow T_{f(p)}\mathbb{R} \cong \mathbb{R} : \text{push forward}$$

then $df_p^* : T^*_{f(p)}\mathbb{R} \longrightarrow T^*_p M$

$w \longmapsto df_p^*(w)$ defined by

$$df_p^*(w) : T_p M \longrightarrow T_{f(p)}\mathbb{R}$$
$$v \longmapsto \left(df_p^*(w)\right)(v) := w\left(df_p(v)\right).$$

Putting these definitions together, we have: