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Veronese Surface

1.a) Let $\mathbb{V}$ be the vector space of real symmetric three-by-three matrices with trace zero. Let (x_{ij}), $1 \leq i, j, \leq 3$ be coordinates on $Mat_3(\mathbb{R})$. Then $\mathbb{V}$ is the zero set of the linear equations:

$$x_{11} + x_{22} + x_{33} = 0$$
$$x_{12} - x_{21} = 0$$
$$x_{13} - x_{31} = 0$$
$$x_{23} - x_{32} = 0$$

which are independant. Therefore $\mathbb{V}$ is $9 - 4 = 5$ dimensional.

1.b) The group $SO(3)$ acts on $\mathbb{V}$ by conjugation: $g \cdot M = gMg^t, g \in SO(3)$, $M \in \mathbb{V}$. Endow $\mathbb{V}$ with the inner product

$$\langle M, S \rangle = tr(MS); M, S \in \mathbb{V}$$

The action of $SO(3)$ on $\mathbb{V}$ is smooth because the action is polynomial in the coordinates (x_{ij}). The action is by isometries because trace is invariant under conjugation

$$\langle gM, gS \rangle = tr(gMSg^t) = tr(MS) = \langle M, S \rangle$$

1.c) Consider $S^4 \subset \mathbb{V}$, $S^4 = \{M \in \mathbb{V} : \langle M, M \rangle = 1\}$. Because $SO(3)$ acts by isometries, we may restrict the action of $SO(3)$ on $\mathbb{V}$ to S^4. To determine the orbits of the action of $SO(3)$ on S^4, we consider the eigenvalue decomposition of symmetric matrices. Any real symmetric matrix may be diagonalized by an element of $SO(3)$. Hence for every $M \in S^4$, there is a $g \in SO(3)$ such that $g \cdot M = gMg^t = D$ where D is a diagonal matrix. Therefore every orbit of the action of $SO(3)$ on S^4 contains a diagonal matrix, and to determine the orbits of $SO(3)$, we may just consider $SO(3) \cdot D$ for diagonal matrices $D \in S^4$.

Now we observe that since the action of $SO(3)$ on S^4 is by conjugation, all the matrices in the same orbit will have the same eigenvalues. This means in particular, for diagonal matrices, that two diagonal matrices can only possibly be in the same orbit if their diagonal entries differ by a permutation. Furthermore, for a diagonal matrix D, all the possible permutations of the diagonal entries are realized by the action of $SO(3)$ since the symmetric group S_3 acts on the diagonal entries of D via conjugation by elements of $SO(3)$. That is, if we use the usual cycle notation for the symmetric group, and let $1 = d_{11}, 2 = d_{22}, 3 = d_{33}$ be the diagonal entries of D, S_3 acts on D via the following elements of $SO(3)$:

$$(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} (12) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} (13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, an orbit of $SO(3)$ is completely determined by specifying the eigenvalues because each orbit will contain all the diagonal matrices with those specified eigenvalues. We conclude that there are two possible orbit types. Orbits with 3 distinct eigenvalues and orbits with 2 distinct eigenvalues (orbits with only 1 eigenvalue are ruled out because the only traceless diagonal matrix with exactly 1-eigenvalue is 0 and $0 \notin S^4$).

first half of 2.c)&1.d) First we consider when the orbit has 2 distinct eigenvalues. Then one of the diagonal matrices in the orbit will be of the form

$$D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{11} & 0 \\ 0 & 0 & -2d_{11} \end{pmatrix}$$

Since $\langle D, D \rangle = 1$, we may solve for $d_{11} = \pm\frac{1}{\sqrt{6}}$. Hence there are exactly two orbits of $SO(3)$ with two distinct eigenvalues, and each of the orbits contains exactly three diagonal matrices given by the three possible permutations of the diagonal elements of D.

1.c) cont. To determine the geometric nature of the orbits, we calculate the isotropy group of D. Let $H \subset SO(3)$ denote the isotropy group of D. Then $g = (a_{ij}) \in H$ if and only if $gH = Hg$. That is, if and only if

$$\begin{pmatrix} d_{11}a_{11} & d_{11}a_{12} & -2d_{11}a_{13} \\ d_{11}a_{21} & d_{11}a_{22} & -2d_{11}a_{23} \\ d_{11}a_{31} & d_{11}a_{32} & -2d_{11}a_{33} \end{pmatrix} = \begin{pmatrix} d_{11}a_{11} & d_{11}a_{12} & d_{11}a_{13} \\ d_{11}a_{21} & d_{11}a_{22} & d_{11}a_{23} \\ -2d_{11}a_{31} & -2d_{11}a_{32} & -2d_{11}a_{33} \end{pmatrix}$$

Hence $a_{13} = a_{31} = a_{23} = a_{32} = 0$, which implies $a_{33} = \pm 1$. If $a_{33} = \pm 1$, in order for $g \in SO(3)$, it must be the case that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \pm\cos(\theta) & \sin(\theta) \\ \mp\sin(\theta) & \cos(\theta) \end{pmatrix}$$

The matrices in the upper left corner of g clearly hint that $H \cong O(2)$, and since $-1 \cdot -1 = 1$ and $-1 \cdot 1 = -1$ the ± 1 in the bottom right corner is compatible with the multiplication of the upper left corner in $O(2)$. Therefore $H \cong O(2)$. We've shown that the 2 orbits of $SO(3)$ with 2 distinct eigenvalues are diffeomorphic to $SO(3)/O(2)$. Since $O(2)$ is 1-dimensional, and $SO(3)$ is 3 dimensional, these orbits have dimension $3 - 1 = 2$.

1.c)cont.& second half of 2.c) We now consider the case when the orbit has 3 distinct eigenvalues. Then the diagonal matrices in the orbit will be of the form

$$D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$$

where d_{11}, d_{22}, d_{33} are pairwise distinct and $d_{11} + d_{22} + d_{33} = 0$. In this case there are six diagonal matrices in the orbit of D corresponding to the six permutations of the diagonal entries of D.

To determine the geometric nature of the orbit, we once again calculate the isotropy group of D. Let H be the isotropy group of D, and let $g = (a_{ij}) \in SO(3)$. Then $g \in H$ if and only if $gD = Dg$, that is if and only if

$$\begin{pmatrix} d_{11}a_{11} & d_{22}a_{12} & d_{33}a_{13} \\ d_{11}a_{21} & d_{22}a_{22} & d_{33}a_{23} \\ d_{11}a_{31} & d_{22}a_{32} & d_{33}a_{33} \end{pmatrix} = \begin{pmatrix} d_{11}a_{11} & d_{11}a_{12} & d_{11}a_{13} \\ d_{22}a_{21} & d_{22}a_{22} & d_{22}a_{23} \\ d_{33}a_{31} & d_{33}a_{32} & d_{33}a_{33} \end{pmatrix}$$

Since d_{11}, d_{22}, d_{33} are distinct we conclude that g must be a diagonal matrix. The set of diagonal matrices in $SO(3)$ is a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein-four group) given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and each of these matrices fixes D under conjugation. Hence $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, so H is discrete and has dimension 0. Therefore, in this case, the orbit is diffeomorphic to $SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and has dimension $3 - 0 = 3$.

The 2 dimensional orbits are called *Veronese surfaces*. Next, we show that a Veronese surface is an embedded copy of $\mathbb{RP}^2$ in S^4.

1.e) Since the 2 dimensional orbits are diffeomorphic to $SO(3)/O(2)$, to show that the Veronese surfaces are embedded copies of $\mathbb{RP}^2$, we show that $SO(3)/O(2)$ is diffeomorphic to $\mathbb{RP}^2$. Consider the action of $SO(3)$ on $\mathbb{RP}^2$ given by left multiplication: $g \cdot [x : y : z] = [g(x, y, z)]$. The action is well defined since

$$g \cdot [x : y : z] = [g(x, y, z)] = [\lambda g(x, y, z)] = [g(\lambda x, \lambda y, \lambda z)] = g \cdot [\lambda x : \lambda y : \lambda z]$$

Furthermore, the action factors through the covering map $\pi : S^2 \to \mathbb{RP}^2$ and the action of $SO(3)$ on S^2 by left multiplication, since for $(x, y, z) \in S^2$,

$$\pi(g(x, y, z)) = [g(x, y, z)] = g \cdot [x : y : z] = g \cdot \pi(x, y, z)$$

Since the action of $SO(3)$ on S^2 is transitive the action of $SO(3)$ on $\mathbb{RP}^2$ is transitive. Letting $[0 : 0 : 1] \in \mathbb{RP}^2$, $\mathbb{RP}^2$ is diffeomorphic to $SO(3)$ modulo the isotropy group of

$[0:0:1]$. $(0,0,1)$ and $(0,0,-1)$ map to $[0:0:1]$ under the covering map $S^2 \to \mathbb{RP}^2$. Therefore the isotropy group, H of $[0:0:1]$ is given by

$$\begin{aligned} H &= \{g \in SO(3) : g \cdot [0:0:1] = [0:0:1]\} \\ &= \{g \in SO(3) : g \cdot \pi(0,0,1) = [0:0:1]\} \\ &= \{g \in SO(3) : \pi(g(0,0,1)) = [0:0:1]\} \\ &= \{g \in SO(3) : g(0,0,1) = (0,0,1) \text{ or } g(0,0,1) = (0,0,-1)\} \\ &= \left\{ \begin{pmatrix} \pm\cos(\theta) & \sin(\theta) & 0 \\ \mp\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} : \theta \in [0, 2\pi] \right\} \\ &\cong O(2) \end{aligned}$$

Hence $SO(3)/O(2) \cong \mathbb{RP}^2$.

1.f) We can now use the fact that $\mathbb{RP}^2$ is not orientable, while S^4 is orientable to prove that the normal bundle to a Veronese surface in S^4 is not orientable. Let $\mathbb{RP}^2 \subset S^4$ be the Veronese surface in S^4, let $N\mathbb{RP}^2$ be the normal bundle to $\mathbb{RP}^2$ in S^4, and let $T\mathbb{RP}^2$ be the tangent bundle to $\mathbb{RP}^2$ in S^4. By definition of normal bundle, we have $TS^4 = T\mathbb{RP}^2 \oplus N\mathbb{RP}^2$ where TS^4 is the tangent bundle of S^4 with $\pi : TS^4 \to S^4$. Since $\mathbb{RP}^2$ is not orientable, $T\mathbb{RP}^2$ is not an orientable bundle, and since S^4 is orientable, TS^4 is an orientable bundle.

By way of contradiction, suppose $N\mathbb{RP}^2$ is orientable. Fix a bundle orientation for TS^4 which induces a bundle orientation on $N\mathbb{RP}^2 \subset TS^4$. A bundle orientation is by definition, a covering by local trivializations where all the transition functions have positive determinant. Let $\{U_\alpha, \varphi_\alpha\}$ be the bundle orientation of TS^4. Since $TS^4 = T\mathbb{RP}^2 \oplus N\mathbb{RP}^2$ we may choose $\{U_\alpha, \varphi_\alpha\}$ such that $\{U_\alpha \cap \mathbb{RP}^2, \varphi_\alpha\}$ is a covering of $\mathbb{RP}^2$ by local trivializations for $T\mathbb{RP}^2$ and $N\mathbb{RP}^2$ when φ_α is restricted to the correct subset of TS^4. Since $T\mathbb{RP}^2$ is not an orientable bundle, there exists $p \in T\mathbb{RP}^2 \subset TS^4$ and $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$, $p \in \pi^{-1}(U_\alpha \cap U_\beta)$ such that the transition function $g_{\alpha\beta}|_{T\mathbb{RP}^2}$ restricted to $T\mathbb{RP}^2$ has negative determinant at p. Since TS^4 and $N\mathbb{RP}^2$ are oriented, $g_{\alpha\beta}$ and $g_{\alpha\beta}|_{N\mathbb{RP}^2}$ have positive determinants at p. But

$$g_{\alpha\beta}(p) = \begin{pmatrix} g_{\alpha\beta}|_{T\mathbb{RP}^2}(p) & 0 \\ 0 & g_{\alpha\beta}|_{N\mathbb{RP}^2}(p) \end{pmatrix}$$

so $\det(g_{\alpha\beta}(p)) = \det(g_{\alpha\beta}|_{T\mathbb{RP}^2}(p)) \det(g_{\alpha\beta}|_{N\mathbb{RP}^2}(p)) < 0$ and we have a contradiction.

2. Now we let Γ be the set of all traceless diagonal matrices on S^4. We know Γ intersects every $SO(3)$ orbit. In particular, we know Γ intersects the Veronese surfaces in exactly three places and Γ intersects the 3-dimensional orbits in exactly six places.

2.a)&e) Let $\mathbb{D} \subset Mat_3(\mathbb{R})$ be the set of all diagonal matrices, and let $\mathbb{W} = \mathbb{D} \cap \mathbb{V}$, so $\mathbb{W}$ is a 2 dimensional subspace of $\mathbb{V}$ given by the set

$$\mathbb{W} = \left\{ \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & -d_{11} - d_{22} \end{pmatrix} \right\}$$

An orthonormal basis for $\mathbb{W}$ is given by

$$e_1 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} e_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\Gamma = \mathbb{D} \cap S^4 = \mathbb{W} \cap S^4$. Then

$$\Gamma = \{v = ae_1 + be_2 : \langle v, v \rangle = a^2 + b^2 = 1\} = \{\cos(\theta)e_1 + \sin(\theta)e_2 : \theta \in [0, 2\pi]\}$$

so Γ is a circle, which may be paramtrized by the unit speed curve

$$\begin{aligned} C : \quad S^1 &\longrightarrow \Gamma \\ \theta &\longmapsto c(\theta)e_1 + s(\theta)e_2 \end{aligned}$$

where $c(\theta) = \cos(\theta)$ and $s(\theta) = \sin(\theta)$. For $D \in \Gamma$, $D = C(\theta)$, under the canonical isomorphism $T_D\mathbb{W} = \mathbb{W}$, we have $T_D\Gamma = span(-s(\theta)e_1 + c(\theta)e_2) \subset \mathbb{W}$.

2.b) Now we show that Γ is orthogonal to each $SO(3)$ orbit. Let $D \in \Gamma$, and let $orb(D)$ denote the orbit of D. To calculate $T_D orb(D)$, we consider $\mathbb{V}, SO(3) \subset Mat_3(\mathbb{R})$, and the map:

$$\begin{aligned} \phi : \quad SO(3) &\longrightarrow S^4 \subset \mathbb{V} \\ g &\longmapsto gDg^t \end{aligned}$$

$T_D orb(D) = \phi_*(T_{id}SO(3))$, and $T_{id}SO(3) = \{A \in Mat_3(\mathbb{R}) : A + A^t = 0\} \subset Mat_3(\mathbb{R})$. We may represent $A \in T_{id}SO(3)$ by the curve $\gamma : (-\epsilon, \epsilon) \to Mat_3(\mathbb{R})$, $\gamma(t) = id + tA$, so $\phi \circ \gamma : (-\epsilon, \epsilon) \to Mat_3(\mathbb{R})$ is a curve in $Mat_3(\mathbb{R})$ tangent to $orb(D)$. A general element of $T_D orb(D)$ is then of the form

$$\begin{aligned} \phi_*(A) &= [\phi \circ \gamma] \\ &= \frac{d}{dt}\bigg|_{t=0} (id + tA)D(id + tA)^t \\ &= \frac{d}{dt}\bigg|_{t=0} (D + t(DA^t + AD) + t^2 ADA^t) = DA^t + AD \end{aligned}$$

Observe that A and A^t have zeros on the diagonal and D is a diagonal matrix. Therefore DA^t and AD will have zeros on the diagonal. It is also clear that $DA^t + AD$ is symmetric, so $DA^t + AD \in \mathbb{V}$. Hence we have $T_D orb(D) = \{DA^t + AD : A + A^T = 0\} \subset \mathbb{V}$.

We need to show that $T_D\Gamma$ is orthogonal to $T_D orb(D)$. To do this, we extend e_1, e_2 to an orthonormal basis of $\mathbb{V}$:

$$e_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

Under the canonical isomorphism $T_D\mathbb{V} = \mathbb{V}$, $T_D\Gamma \subset span(e_1, e_2)$ and every element of $T_D orb(D)$ has zeros on the diagonal. Therefore $T_D orb(D) \subset span(e_3, e_4, e_4)$. Hence $T_D\Gamma$ and $T_D orb(D)$ are orthogonal.

2.d) Now we consider the manifold $S^4/SO(3)$, and show that $S^4/SO(3)$ is homeomorphic to a closed interval whose two endpoints correspond to the two Veronese curves. $S^4/SO(3)$ is the topological space whose points are equivalence classes of points in S^4 under the equivalence relation $M \sim M'$ if and only if there exists $g \in SO(3)$ such that $gMg^t = M'$. Let $[M]$ denote the equivalence class in $S^4/SO(3)$ represented by M.

The projection map $\pi : S^4 \to S^4/SO(3)$, $\pi(M) = [M]$ is a continuous open map. We also have the inclusion map $\iota : \Gamma \to S^4$. Since Γ intersects every orbit of $SO(3)$, $\pi \circ \iota : \Gamma \to S^4/SO(3)$ is continuous and surjective. Hence $S^4/SO(3) = \pi \circ \iota(\Gamma) = \Gamma/SO(3)$. Recall that for diagonal matrices $D, D' \in \Gamma$ and $g \in SO(3)$, $gDg^t = D'$ if and only if $g \in S_3 \subset SO(3)$ where S_3 acts by permuting the diagonal entries of D. Therefore $S^4/SO(3) = \Gamma/SO(3) = \Gamma/S_3$. Hence if we determine Γ/S_3 topologically, we determine $S^4/SO(3)$ topologically.

S_3 acts linearly on the set of diagonal matrices $\mathbb{D} \subset Mat_3(\mathbb{R})$ by permuting the diagonal entries, and S_3 fixes the 2-dimensional subspace $\mathbb{W}$ of traceless diagonal matrices. To determine the action of S_3 on Γ, we write down the action of S_3 on $\mathbb{W}$ with respect to the basis $e_1, 2_2$. (123) and (12) (using the notation from before) generate S_3. With respect to the basis e_1, e_2, (123) acts on $\mathbb{W}$ in the following way:

$$(123) \cdot e_1 = \begin{pmatrix} -\frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{pmatrix} = -\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2$$

$$(123) \cdot e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\sqrt{3}}{2}e_1 - \frac{1}{2}e_2$$

Therefore as a linear map on $\mathbb{W}$, $(123) = \begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ -\sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{pmatrix}$ is rotation by $\frac{2\pi}{3}$. With respect to e_1, e_2, (12) acts on $\mathbb{W}$ by:

$$(12) \cdot e_1 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} = e_1$$

$$(123) \cdot e_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} = -e_2$$

Therefore as a linear map on $\mathbb{W}$, $(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is reflection about the line generated by e_1. We see that S_3 is acting on Γ as the symmetry group of the triangle with vertices $C(0), C(\frac{2\pi}{3}), C(\frac{4\pi}{3})$. Therefore Γ/S_3 is homeomorphic to a closed interval with endpoints $C(0)$ and $C(\frac{\pi}{3})$. We have

$$C(0) = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

$$C(\tfrac{\pi}{3}) = c(\tfrac{\pi}{3})e_1 + s(\tfrac{\pi}{3})e_2 = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

so $C(0)$ is on one Veronese surface, while $C(\frac{\pi}{3})$ is on the other Veronese surface. Therefore Γ/S_3 is homeomorphic to an interval with endpoints the two Veronese surfaces.

2.f)&g) Now we consider the map $\Psi : SO(3) \times S^1 \to S^4$ given by $\Psi(g, \theta) \mapsto g \cdot C(\theta)$, and compute the pullback of the standard metric ds^2 on S^4 (induced from the inner product on $\mathbb{V}$) in terms of the standard basis for left-invariant one-forms on $SO(3)$. Since ds^2 is invariant under the action of $SO(3)$, we have

$$\begin{aligned}(\Psi^* ds^2)_{(g,\theta)}(A,B) &= ds^2_{(g\cdot C(\theta))}(\Psi_* A, \Psi_* B)\\ &= ds^2_{g^t g\cdot C(\theta)}(g^t\Psi_* A, g^t \Psi_* B)\\ &= ds^2_{(id,C(\theta))}(\Psi_* A,\Psi_* B) = (\Psi^* ds^2)_{(id,\theta)}(A,B)\end{aligned}$$

Therefore to calculate $\Psi^* ds^2$, it suffices to calculate at $(id,\theta)\in SO(3)\times S^1$. Let

$$E_1 = \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} E_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} E_3 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}$$

be a basis for $T_{id}SO(3)$. Let ∂_θ be a basis for $T_\theta S^1$. Let $\gamma_i = (id,\theta)+t(E_i,0)$ be a curve in $Mat_3(\mathbb{R})\times S^1$ tangent to $(id,\theta)\in SO(3)\times S^1$. Then we have

$$\begin{aligned}\Psi_*(E_1) &= [\Psi\circ\gamma_1]\\ &= \frac{d}{dt}\bigg|_{t=0}\Psi(\gamma(t))\\ &= \frac{d}{dt}\bigg|_{t=0}\begin{pmatrix} 1 & t & 0\\ -t & 1 & 0\\ 0 & 0 & 1\end{pmatrix} C(\theta)\begin{pmatrix} 1 & -t & 0\\ t & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix} 0 & -\sqrt{2}\sin(\theta) & 0\\ -\sqrt{2}\sin(\theta) & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

Similar calculations yield

$$\begin{aligned}\Psi_*(E_2) &= \begin{pmatrix} 0 & 0 & -\frac{3}{\sqrt{6}}\cos(\theta)-\frac{1}{\sqrt{2}}\sin(\theta)\\ 0 & 0 & 0\\ -\frac{3}{\sqrt{6}}\cos(\theta)-\frac{1}{\sqrt{2}}\sin(\theta) & 0 & 0\end{pmatrix}\\ \Psi_*(E_3) &= \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -\frac{3}{\sqrt{6}}\cos(\theta)+\frac{1}{\sqrt{2}}\sin(\theta)\\ 0 & -\frac{3}{\sqrt{6}}\cos(\theta)+\frac{1}{\sqrt{2}}\sin(\theta) & 0\end{pmatrix}\\ \Psi_*(\partial_\theta) &= C'(\theta)\end{aligned}$$

Then we have

$$\begin{aligned}
&\Psi^* ds^2(E_1, E_1) = \langle \Psi_*(E_1), \Psi_*(E_1)\rangle = 4\sin^2(\theta) \\
&\Psi^* ds^2(E_2, E_2) = \langle \Psi_*(E_2), \Psi_*(E_2)\rangle = (\sqrt{3}\cos(\theta) + \sin(\theta))^2 \\
&\Psi^* ds^2(E_3, E_3) = \langle \Psi_*(E_3), \Psi_*(E_3)\rangle = (\sqrt{3}\cos(\theta) - \sin(\theta))^2 \\
&\Psi^* ds^2(\partial_\theta, \partial_\theta) = \langle C'(\theta), C'(\theta)\rangle = 1 \\
&\Psi^* ds^2(E_i, E_j) = 0, i \neq j \\
&\Psi^* ds^2(E_i, \partial_\theta) = 0, \forall i
\end{aligned}$$

Letting $\sigma_1, \sigma_2, \sigma_3$ be the dual left invariant coframe to E_1, E_2, E_3, we then have

$$\Psi^* ds^2 = d\theta^2 + 4\sin^2(\theta)\sigma_1^2 + (\sqrt{3}\cos(\theta) + \sin(\theta))^2\sigma_2^2 + (\sqrt{3}\cos(\theta) - \sin(\theta))^2\sigma_3^2$$

Let $a(\theta) = 2\sin(\theta), b(\theta) = \sqrt{3}\cos(\theta) + \sin(\theta)$ and $c(\theta) = \sqrt{3}\cos(\theta) - \sin(\theta)$ (we will no longer right $c(\theta)$ for $\cos(\theta)$).

2.h) We will now use $a(\theta), b(\theta)$, and $c(\theta)$ to calculate the three-dimensional volume of the $SO(3)$-orbit through $C(\theta) \in \Gamma$. Note that since the Veronese surfaces are 2-dimensional, their 3-dimensional volume is zero. We will calculate the 3-dimensional volume for a generic 3-dimensional orbit represented by $C(\theta)$. We wrote $\Psi^* ds^2$ as the sum of squares, so a volume form on $SO(3)$ is given by

$$a(\theta)\sigma_1 \wedge b(\theta)\sigma_2 \wedge c(\theta)\sigma_3 = \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3$$

$\tilde{\sigma}_1 = a(\theta)\sigma_1, \tilde{\sigma}_2 = b(\theta)\sigma_2, \tilde{\sigma}_3 = c(\theta)\sigma_3$ are an orthonormal coframe on $SO(3)$ with respect to $\Psi^* ds^2$. Recall that for a generic 3-dimensional orbit, the isotropy group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and so we have a $4:1$ covering map $SO(3) \to SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2) = orb(C(\theta))$. Hence

$$\mathrm{Vol}(orb(C(\theta))) = \frac{1}{4}\mathrm{Vol}(SO(3))$$

and

$$\mathrm{Vol}(SO(3)) = \int_{SO(3)} \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3$$

The rest of the argument I am going to make, I'm not sure is correct. Maybe you could point out where some of the misunderstanding is.

To calculate the volume of $SO(3)$ we use the $2:1$ covering map $SU(2) \to SO(3)$ and the fact that $SU(2)$ is diffeomorphic to S^3. Represent $SU(2)$ be matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1$$

We then have $\mathfrak{su}_2$, the Lie algebra of $SU(2)$, a three dimensional real vector space with basis

$$A_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

A_1, A_2, A_3 then generate a frame for $SU(2)$. Let a_1, a_2, a_3 be the dual coframe. We may represent $\pi : SU(2) \to SO(3)$ in the following way, letting $\mathfrak{su}_2 = \mathbb{R}^3$

$$\begin{array}{rcccccc} \pi : & SU(2) & \longrightarrow & SO(3) & & & \\ & A & \longmapsto & \phi_A : & \mathbb{R}^3 = \mathfrak{su}_2 & \longrightarrow & \mathbb{R}^3 = \mathfrak{su}_2 \\ & & & & U & \longmapsto & AUA^{-1} \end{array}$$

Now with E_1, E_2, E_3 as before (the standard basis for $\mathfrak{so}_3 = T_{id}SO(3)$), we have (by a calculation that $d\pi_{id}(M)(U) = [M, U] = MU - UM$ for $M \in \mathfrak{su}_2$, $U \in \mathbb{R}^3 = \mathfrak{su}_2$, $d\pi_{id}(M) \in \mathfrak{so}_3$)

$$\begin{aligned} d\pi_{id}(A_1) &= -2E_1 \\ d\pi_{id}(A_2) &= 2E_2 \\ d\pi_{id}(A_3) &= -2E_3 \end{aligned}$$

Then the 3-form $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ will pull back under π to $-2a_1 \wedge 2a_2 \wedge -2a_3 = 8a_1 \wedge a_2 \wedge a_3$ on $SU(2)$. Since π is a $2:1$ cover, we then have

$$\int_{SO(3)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \frac{1}{2} \int_{SU(2)} 8a_1 \wedge a_2 \wedge a_3 = 4 \int_{SU(2)} a_1 \wedge a_2 \wedge a_3$$

Now to calculate $\int_{SU(2)} a_1 \wedge a_2 \wedge a_3$, we identify $SU(2)$ with S^3 in the following way

$$\begin{array}{rccc} \Phi : & SU(2) & \longrightarrow & S^3 \subset \mathbb{R}^4 \\ & \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} & \longmapsto & (Re(\alpha), Im(\alpha), Re(\beta), Im(\beta)) \end{array}$$

Since Φ is linear, we have $d\Phi_{id} = \Phi$, so

$$\begin{array}{rccc} d\Phi_{id} = \Phi : & \mathfrak{su}_2 & \longrightarrow & T_{\Phi(id)}S^3 \subset \mathbb{R}^4 \\ & A_1 & \longmapsto & (0, 1, 0, 0) \\ & A_2 & \longmapsto & (0, 0, 1, 0) \\ & A_3 & \longmapsto & (0, 0, 0, 1) \end{array}$$

Therefore when we pull back the standard inner product on S^3 induced by the standard inner product in $\mathbb{R}^4$, we get that A_1, A_2, A_3 generates an orthonormal frame for $\mathfrak{su}_2$ in the pulled-back metric from $S^3 \subset \mathbb{R}^4$. Hence, letting dA be the volume form on $S^3 \subset \mathbb{R}^4$, we have

$$2\pi^2 = \int_{S^3} dA = \int_{SU(2)} a_1 \wedge a_2 \wedge a_3$$

Therefore, putting everything we have together we get

$$\begin{aligned}
\mathrm{Vol}(orb(C(\theta))) &= \frac{1}{4}\int_{SO(3)} \tilde{\sigma_1} \wedge \tilde{\sigma_2} \wedge \tilde{\sigma_3} \\
&= \frac{a(\theta)b(\theta)c(\theta)}{4}\int_{SO(3)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\
&= \frac{a(\theta)b(\theta)c(\theta)}{4 \cdot 2}\int_{SU(2)} 8a_1 \wedge a_2 \wedge a_3 \\
&= a(\theta)b(\theta)c(\theta)\int_{SU(2)} a_1 \wedge a_2 \wedge a_3 \\
&= a(\theta)b(\theta)c(\theta)2\pi^2 \\
&= 4\pi^2 \sin(\theta)(\sqrt{3}\cos(\theta) + \sin(\theta))(\sqrt{3}\cos(\theta) - \sin(\theta))
\end{aligned}$$