Manifolds 2 Final

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Veronese surface

1. Let $\mathbb{V}$ be the vector space of real symmetric 3×3 matrices with trace zero.

a). What is the dimension of $\mathbb{V}$?

Proof. Given the following collection of matrices

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$e_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad e_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

we claim that $\{e_1, e_2, e_3, e_4, e_5\}$ forms a basis for $\mathbb{V}$. First, given scalars $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}$ such that

$$\sum_{k=1}^{5} v_k e_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

it is clear that it must be the case that $v_1 = v_2 = v_3 = v_4 = v_5 = 0$. Thus the collection $\{e_1, e_2, e_3, e_4, e_5\}$ is linearly independent. Now any $M \in \mathbb{V}$, for $a, b, c, d, f \in \mathbb{R}$ is of the form

$$M = \begin{bmatrix} a & c & d \\ c & b & f \\ d & f & -a-b \end{bmatrix} = ce_1 + de_2 + fe_3 + ae_4 + be_5.$$

This shows that $\mathbb{V} \subseteq \text{Span}(e_1, e_2, e_3, e_4, e_5)$. Also, for each $1 \leq k \leq 5$, e_k is traceless and symmetric, hence $\text{Span}(e_1, e_2, e_3, e_4, e_5) \subseteq \mathbb{V}$. Thus $\{e_1, e_2, e_3, e_4, e_5\}$ is a basis for $\mathbb{V}$, and so the dimension of $\mathbb{V}$ is 5. □

The group $SO(3)$ acts on $\mathbb{V}$ by conjugation: $g \cdot M = gMg^T$, $g \in SO(3)$, $M \in \mathbb{V}$. Endow $\mathbb{V}$ with the inner product

$$\langle M, S \rangle = \text{tr}(MS); \qquad M, S \in \mathbb{V}.$$

b). Show that the $SO(3)$ action on $\mathbb{V}$ is a smooth action by isometries.

Proof. For $g \in SO(3)$ and $M \in \mathbb{V}$, we are given $g \cdot M = gMg^T$. Since the matrix entries of gMg^T will be polynomials in the matrix entries of g and M, it will be the case that the $SO(3)$ action on $\mathbb{V}$ is smooth.

For each $g \in SO(3)$, define $F_g : \mathbb{V} \to \mathbb{V}$ by $F_g(M) = g \cdot M$. Given $M, N \in \mathbb{V}$ and $g \in SO(3)$,

$$\langle F_g(M), F_g(N) \rangle = \mathrm{tr}((g \cdot M)(g \cdot N)) = \mathrm{tr}(gMg^T gNg^T).$$

As $g \in SO(3)$, we have that $g^T g = \mathrm{Id}_{3\times 3}$, hence $\mathrm{tr}(gMg^T gNg^T) = \mathrm{tr}(gMNg^T)$. A result from linear algebra states that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for two matrices A, B. Hence,

$$\langle F_g(M), F_g(N) \rangle = \mathrm{tr}(gMNg^T) = \mathrm{tr}(MNg^T g) = \mathrm{tr}(MN) = \langle M, N \rangle.$$

Thus for every $g \in SO(3)$, F_g is an isometry on $\mathbb{V}$. Thus the action of $SO(3)$ on $\mathbb{V}$ is a smooth action by isometries. □

Restrict the $SO(3)$ action to the 4–sphere, $S^4 \subset \mathbb{V}$.

c). Use the eigenvalue decomposition of symmetric matrices to show that the orbits of the restricted action fall into two types: orbits with dimension 3 and orbits with dimension 2. For each of the two types of orbits find the closed subgroup $H \subset SO(3)$ such that the orbit is isomorphic to $SO(3)/H$.

Proof. Given a matrix, $M \in S^4$, due to the spectral theorem for real symmetric matrices, there is a $Q \in SO(3)$ such that $D = QMQ^T$ is a diagonal matrix with the eigenvalues of M on the diagonal. Also,

$$\langle D, D \rangle = \mathrm{tr}(D^2) = \mathrm{tr}(QM^2Q^T) = \mathrm{tr}(M^2) = 1,$$

so D is also on S^4. This tells us that any $M \in S^4$ is in the orbit of some diagonal matrix, also in S^4, hence it suffices to verify the claim for the orbits of diagonal matrices in S^4.

Define a map $F_D : SO(3) \to S^4$ by $F_D(g) = g \cdot D = gDg^T$. The computation above shows that the trace of a matrix is invariant under conjugation by elements in $SO(3)$, thus the range of F_D is contained in S^4. (In other words, the action of $SO(3)$ on $\mathbb{V}$ does restrict to $S^4 \subset \mathbb{V}$.) As stated in the previous part, the action of conjugation by elements of $SO(3)$ will be smooth on S^4. This also implies that F_D is a smooth map.

As $SO(3)$ is a Lie group, $SO(3)$ acting on itself by left multiplication is smooth. Also, for any $p, q \in SO(3)$, there exists $qp^{-1} \in SO(3)$ such that $qp^{-1}(p) = q$, i.e. left multiplication is a transitive action on $SO(3)$. Given $g, h \in SO(3)$,

$$F_D(gh) = ghDh^T g^T = g \cdot F_D(h),$$

shows that F_D is equivariant with respect to the actions on $SO(3)$ and S^4. For a fixed $g \in SO(3)$ we can define a map $\alpha_g : SO(3) \to SO(3)$ via left multiplication, i.e. $\alpha_g(h) = gh$. Similarly, for a fixed $g \in SO(3)$, we can define a map $\beta_g : S^4 \to S^4$ via the group action of $SO(3)$ on S^4, i.e. $\beta_g(M) = g \cdot M$. The maps α_g, β_g are diffeomorphisms for any value of $g \in SO(3)$ as $\alpha_{g^T}, \beta_{g^T}$ are the inverse maps, respectively, and all the maps are smooth due to the actions of $SO(3)$ on itself and S^4 being smooth. Due to the equivariance of F_D we have $\beta_g \circ F_D = F_D \circ \alpha_g$, and as $d(F \circ G) = dF \circ dG$ in general, we have that the following diagram commutes,

$$\begin{array}{ccc} T_pSO(3) & \xrightarrow{(dF_D)_p} & T_{F_D(p)}S^4 \\ \downarrow (d\alpha_g)_p & & \downarrow (d\beta_g)_p \\ T_{gp}SO(3) & \xrightarrow{(dF_D)_{gp}} & T_{g\cdot F_D(p)}S^4 \end{array}$$

Since left multiplication on $SO(3)$ is a transitive action, we have that F_D must be of constant rank.

To find the dimension of the orbit $O_D = F_D(SO(3))$, by what was shown above, it suffices to check the rank of dF_D at any point of $SO(3)$. For the sake of simplicity, we check at the identity of $SO(3)$. We know that $T_ISO(3)$ is the space of all skew–symmetric matrices from a previous result. Given $h \in T_ISO(3)$, the curve $\gamma(t) = I + th$ is a line in the space of 3×3 matrices that is 'close' to lying in $SO(3)$ for small t with $\gamma'(0) = h$. Hence,

$$\begin{aligned} (dF_D)_I(h) &= \frac{d}{dt}(F_D \circ \gamma(t))\Big|_{t=0} = \frac{d}{dt}(D + tDh^T + thD + t^2hDh^T)\Big|_{t=0} \\ &= (Dh^T + hD + 2thDh^T)\Big|_{t=0} = Dh^T + hD. \end{aligned}$$

Denoting $\lambda_1, \lambda_2, \lambda_3$ as the entries on the diagonal of D, we have the following, as $D \in S^4$ and $\mathrm{tr}(D) = 0$.

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 0 \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= 1 \end{aligned}$$

By the rank–nullity theorem, to find the rank of $(dF_D)_I$, it suffices to find the kernel of $(dF_D)_I$, i.e. the $h \in T_ISO(3)$ such that $hD = Dh$ as h is skew–symmetric. Using the general form of a skew–symmetric matrix

$$h = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

for some $a, b, c \in \mathbb{R}$, the equation $hD = Dh$ leads to the following three equations

$$\begin{aligned} a(\lambda_1 - \lambda_2) &= 0 \\ b(\lambda_1 - \lambda_3) &= 0 \\ c(\lambda_2 - \lambda_3) &= 0 \end{aligned}$$

Upon examination, as $\mathrm{tr}(D) = 0$, it can not be the case that $\lambda_1 = \lambda_2 = \lambda_3$. Thus, we have only two cases, mainly one in which two out of three eigenvalues are equal, and one in which all eigenvalues are distinct.

For the D in which all the eiqnvalues are distinct, via our equations above, $a = b = c = 0$, hence $\mathrm{Ker}((dF_D)_I) = \{\vec{0}\}$, where $\vec{0}$ denotes the 3×3 0–matrix. In this case, $\dim(O_D) = 3$.

For the D in which two of the eigenvalues agree, denoting

$$h_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; h_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; h_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

we find that $\mathrm{Ker}((dF_D)_I) = \mathrm{Span}(h_i)$ for $1 \leq i \leq 3$ depending on which two eigenvalues agree. However, in all cases the kernel of $(dF_D)_I$ is of dimension one, hence $\dim(O_D) = 2$.

A well known result from group theory states that a group G modulo the stabilzer of p (the isotropy group of p) is isomorphic to the orbit of p, i.e. $G/G_p \cong O_p$. Denote H_D as the isotropy group of D under the action of $SO(3)$ on S^4. Since F_D is a smooth map and $H_D = F_D^{-1}(\{D\})$, H_D will be the closed group such that $SO(3)/H_D \cong O_D$.

Case 1: The eigenvalues of D are distinct. Take $g \in H_D$, thus $gDg^T = D$ or $gD = Dg$ as $g \in SO(3)$. Taking

$$g = \begin{bmatrix} a & b & c \\ d & e & f \\ A & B & C \end{bmatrix}.$$

The equation $gD = Dg$ implies all the entries of g besides the diagonal are 0. Also, $\det(g) = 1$, $gg^T = \mathrm{Id}_{3\times3}$ as $g \in SO(3)$. This implies $aec = 1$, $a^2 = e^2 = c^2 = 1$. Hence,

$$H_D = \left\{ \mathrm{Id}_{3\times3}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

which is isomorphic to the Klein 4–group.

Case 2: Two of the eigenvalues of D are equal. Using a similar argument to the above, finding $g \in SO(3)$ so that $gD = Dg$ implies that g must be of the form

$$g = \begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & C \end{bmatrix},$$

where $a^2 + b^2 = d^2 + e^2 = C^2 = 1$, $ad + be = 0$, and $C(ae - bd) = 1$. These equations in turn imply that g is of the form of one of these two matrices

$$\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $a^2 + b^2 = 1$. The matrices of the first type are isomorphic to 2×2 matrices of determinant 1. The matrices of the second type are isomorphic to 2×2 matrices of determinant -1. Thus, in this case H_D is isomorphic to $O(2)$. □

d). Show that there are exactly two 2–dimensional orbits. These orbits all called *Veronese surfaces.*

Proof. The 2–dimensional orbits occur as the orbits of a diagonal matrix D in which two of the eigenvalues are equal. Using the notation of the previous problem, the eigenvalues must also satisfy

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 0 \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= 1. \end{aligned}$$

Solving these equations in the cases $\lambda_1 = \lambda_2$, $\lambda_1 = \lambda_3$, and $\lambda_2 = \lambda_3$, leads to the following six matrices.

$$\begin{bmatrix} \pm\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \pm\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \mp\frac{2}{\sqrt{6}} \end{bmatrix}, \quad \begin{bmatrix} \pm\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \mp\frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & \pm\frac{1}{\sqrt{6}} \end{bmatrix}, \quad \begin{bmatrix} \mp\frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & \pm\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \pm\frac{1}{\sqrt{6}} \end{bmatrix}$$

The following matrices are in $SO(3)$ and permute the order in which the eigenvalues appear on the diagonal when acting on D by conjugation.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus, the six matrices above only correspond to two distinct orbits. Thus the two Veronese surfaces correspond to the orbits of the following two diagonal matrices

$$\begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

□

e). Show that a Veronese surface is an embedded copy of $\mathbb{RP}^2$ in S^4.

Proof. Define an action $SO(3) \times S^2 \to S^2$ for $g \in SO(3)$ and $v \in S^2$ by $g \cdot v = gv$. Given any $v, w \in S^2$, the action is transitive. This can be seen intuitively, as the cross product $v \times w$ will give the direction of the axis of rotation and then rotate along this axis an angle equivalent to the angle between v and w in the plane they generate.

Define the antipodal map $F : S^2 \to S^2$, $F(v) = -v$. Clearly, for any $g \in SO(3)$,

$$g \cdot F(v) = g \cdot (-v) = -gv = F(g \cdot v),$$

hence the antipodal map and the action commute.

The space $\mathbb{RP}^2$ can also be defined as $S^2/\sim$ where $\sim$ is the equivalence relation $v \sim -v$. Given $w \in S^2$,

$$g \cdot (-w) = -(g \cdot w),$$

shows that in $\mathbb{RP}^2$, $g \cdot (-w) = g \cdot w$. Thus, the action of $SO(3)$ can be defined on $\mathbb{RP}^2$ as left multiplication since the computation above showed that the action is well-defined. (Independent of representative chosen) The action is transitive on $\mathbb{RP}^2$ as it was transitive on S^2. Hence $\mathbb{RP}^2 \cong SO(3)/H_v$ for any $v \in \mathbb{RP}^2$ where H_v is the isotropy group of v.

Taking $v = (0, 0, 1)$, we must find $g \in SO(3)$ so that

$$\begin{bmatrix} a & b & c \\ d & e & f \\ A & B & C \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}.$$

This equation implies that $c = f = 0$, $C = \pm 1$. As $g \in SO(3)$, we have that $g^T g = \mathrm{Id}_{3\times 3}$, thus

$$\begin{bmatrix} a^2 + b^2 & ad + be & aA + bB \\ ad + be & d^2 + e^2 & dA + eB \\ aA + bB & dA + eB & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $aA + bB = dA + eB = 0$. If $a \neq 0$, then $(ae - bd)B = 0$, and as $\det(g) = 1$ it must be the case that $A = B = 0$. If $a = 0$, then $A = B = 0$ as the condition that $\det(g) = 1$ implies that $b, d \neq 0$.

Thus the entries of g must satisfy the equations; $a^2 + b^2 = d^2 + e^2 = C(ae - bd) = 1$, $ad + be = 0$, and $C = \pm 1$. We found earlier, in part c, that H_v will be isomorphic to $O(2)$. Hence for the diagonal matrix, D, in Case 2 of part c

$$\mathbb{RP}^2 \cong SO(3)/H_v = SO(3)/H_D \cong O_D.$$

Showing that the Veronese surface O_D is an embedded copy of $\mathbb{RP}^2$ in S^4. □

f). Is the normal bundle to a Veronese surface within S^4 an oriented vector bundle? Why or why not?

Proof. Take $D \in S^4$ so that O_D is a Veronese surface. Then both O_D and S^4 have the standard metric induced by the inner product on $\mathbb{V}$. Thus, at any point $p \in O_D$, we may view the normal space, N_pO_D as the orthogonal complement to the tangent space. Thus,

$$T_pO_D \oplus N_pO_p = T_pS^4.$$

If the normal bundle was orientable, then as S^4 is orientable, there would be an induced orientation on O_D. This is a contradiction as it was just shown that O_D is an embedded copy of the non–orientable surface $\mathbb{RP}^2$. □

2. **Veronese continued**

a). Show that the diagonal matrices intersect S^4 in a circle Γ which intersects every $SO(3)$ orbit.

Proof. Call $\mathcal{D}$ the vector space of 3×3 diagonal matrices. Then it is clear that $\mathcal{D}$ is isomorphic to $\mathbb{R}^3$. Intersecting $\mathcal{D}$ with $\mathbb{V}$ gives a two–dimensional subspace of $\mathcal{D}$ as we are losing one degree of freedom due to the restrction on the trace. Using the inner product on $\mathbb{V}$, one can find an orthonormal basis $\{e_1, e_2\}$ for $\mathcal{D} \cap \mathbb{V}$.

$$e_1 = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{bmatrix}, \quad e_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Hence, the set of diagonal matrices in $\mathbb{V}$ lying on S^4 will be $\Gamma := \{xe_1 + ye_2 \mid \|xe_1 + ye_2\| = 1\}$. And

$$\begin{aligned} 1 &= \|xe_1+ye_2\|^2 = \langle xe_1+ye_2, xe_1+ye_2\rangle = \operatorname{tr}\left(\begin{bmatrix} \frac{y}{\sqrt{2}}+\frac{x}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{2x}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{x}{\sqrt{6}}-\frac{y}{\sqrt{2}} \end{bmatrix}^2\right) \\ &= \left(\frac{y}{\sqrt{2}}+\frac{x}{\sqrt{6}}\right)^2 + \left(-\frac{2x}{\sqrt{6}}\right)^2 + \left(\frac{x}{\sqrt{6}}-\frac{y}{\sqrt{2}}\right)^2 = x^2+y^2. \end{aligned}$$

Thus $\Gamma := \{xe_1+ye_2 \mid x^2+y^2 = 1\}$. For any $M \in S^4$, by the spectral theorem for real symmetric matrices, there exists $g \in SO(3)$ such that gMg^T is a diagonal matrix, with $\operatorname{tr}(gMg^T) = \operatorname{tr}(M) = 0$, $\operatorname{tr}(gM^2g^T) = \operatorname{tr}(M^2) = 1$. Thus $gMg^T \in \Gamma$. Thus, as $M \in S^4$ was taken arbitrarily, we have shown that the diagonal matrices intersect S^4 in a circle, Γ, which intersects every $SO(3)$ orbit. □

b). Show that Γ is orthogonal to each $SO(3)$ orbit.

Proof. Each $SO(3)$ orbit will intersect Γ at a diagonal matrix. Use this diagonal matrix, call it D as the representative of the $SO(3)$ orbit. As the space of diagonal matrices, $\mathcal{D}$ is a vector space, we have that $T_X\mathcal{D} \cong \mathcal{D}$ for any $X \in \mathcal{D}$. Hence, $T_D\Gamma \subset T_D\mathcal{D}$ can be thought of as some collection of diagonal matrices.

Now, using the orbit map from problem 1c, $F_D : SO(3) \to S^4$, we have that $(dF_D)_I$ will be onto T_DO_D. Hence for any $x \in T_DO_D$ there exists $h \in T_ISO(3)$ such that $x = (dF_D)_I(h)$. Thus $x = Dh^T + hD$ for some skew–symmetric matrix h.

Now, any skew-symmetric matrix has only zero as its diagonal entries, because of which, the product of a skew–symmetric matrix and a diagonal matrix will have 0 as the only entry on the diagonal. Hence Dh^T and hD have only zero along the diagonal, and thus so will x. This is true of any $x \in T_DO_D$. Thus given any $x \in T_DO_D$ and any $g \in T_D\Gamma$ we have

$$\langle g, x\rangle = \operatorname{tr}(gx) = 0,$$

as the product of a diagonal matrix and a matrix with only zero along the diagonal has only zero along the diagonal. Thus $T_D\Gamma \perp T_DO_D$. □

c). Show that Γ intersects a Veronese surface in exactly three points while it intersects a 3–dimensional orbit in exactly six points.

Proof. Let $M \in S^4$. As we traverse the orbit of M, it is analoguous to viewing M under different bases. As the eigenvalues of a matrix are independent of the basis chosen, the orbit of M will intersect Γ only at diagonal matrices with the eigenvalues as the non-zero entries. As shown in problem 1d, $SO(3)$ acting on S^4 by conjugation contains elements that will permute the elements on the diagonal of a diagonal matrix. Therefore all permutations of a diagonal matrix also intersect Γ if the given diagonal matrix does. Thus the orbit of M will intersect Γ precisely the same number of times as there are orderings of the eigenvalues on the diagonal.

Thus for M in an orbit which is not a Veronese surface, as found earlier, the eigenvalues of M are all distinct. The number of ways to order 3 distinct objects is 6. Thus, the orbit of M will intersect Γ six times.

If M is in an orbit which is a Veronese surface, then two of the eigenvalues of M are equal. The number of orderings of three objects in which two are equal is 3. Thus, the orbit of M will intersect Γ three times. □

d). Show that $S^4/SO(3)$ is topologically a closed interval whose two endpoints correspond to the two Veronese surfaces.

Proof. Making use of parameterization of Γ (below in 2e) we can find the values of θ so that $C(\theta)$ equals one of the six matrices listed in problem 1d. In doing so, we find that the values $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ correspond to elements in an orbit that is a Veronese surface.

For $0 < \theta < \frac{\pi}{3}$, $C(\theta)$ corresponds to the intersection of an orbit that is not a Veronese surface with the circle Γ. For this θ, denote $C(\theta) = g$, and g is of the form

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1.$$

And in the previous proof, part 2c, we found that any other matrix in $\Gamma \cap O_g$ will be a diagonal matrix in which the eigenvalues (diagonal elements) will be permuted.

If $d \in \Gamma \cap O_g$ and d is a matrix formed by an even permutation of the diagonal elements of g, then d is

$$\begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

For d of the first type

$$\begin{aligned}\langle d,g\rangle &= \operatorname{tr}(dg)=\operatorname{tr}\left(\begin{bmatrix}\lambda_1\lambda_2 & 0 & 0\\ 0 & \lambda_2\lambda_3 & 0\\ 0 & 0 & \lambda_1\lambda_3\end{bmatrix}\right)\\ &= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3=-(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)=-\frac{1}{2}.\end{aligned}$$

The last equality is due to the fact that $\operatorname{tr}(g)=0$ and $\operatorname{tr}(g^2)=1$ implies that $\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2=\frac{1}{2}$. Thus, as $g,d\in\Gamma$, $\|g\|=\|d\|=1$, the angle between d,g is $\cos^{-1}(-\frac{1}{2})=\frac{2\pi}{3}$. A similar computation may be performed for matrices of the other type. Thus we have shown that for any $d\in\Gamma\cap O_g$ such that d is a matrix formed by an even permutation of the diagonal elements of g, then the angle between d and g is $\frac{2\pi}{3}$.

The computations above could be done for any $g=C(\theta)$. Hence a similar argument to the one directly above shows that any two elements in $\Gamma\cap O_g$ related by an even permutation have an angle of $\frac{2\pi}{3}$ between them.

Now, briefly returning to our case of $g=C(\theta)$ for $0<\theta<\frac{\pi}{3}$, define $h=C(-\theta)$. Now,

$$g=\cos(\theta)e_1+\sin(\theta)e_2=\begin{bmatrix}\frac{\cos(\theta)}{\sqrt{6}}+\frac{\sin(\theta)}{\sqrt{2}} & 0 & 0\\ 0 & -\frac{2\cos(\theta)}{\sqrt{6}} & 0\\ 0 & 0 & \frac{\cos(\theta)}{\sqrt{6}}-\frac{\sin(\theta)}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}.$$

Hence,

$$h=\cos(-\theta)e_1+\sin(-\theta)e_2=\begin{bmatrix}\frac{\cos(\theta)}{\sqrt{6}}-\frac{\sin(\theta)}{\sqrt{2}} & 0 & 0\\ 0 & -\frac{2\cos(\theta)}{\sqrt{6}} & 0\\ 0 & 0 & \frac{\cos(\theta)}{\sqrt{6}}+\frac{\sin(\theta)}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\lambda_3 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_1\end{bmatrix}.$$

Thus h is the matrix formed by an odd permutation of elements on the diagonal of g. Thus, by what was stated above, any matrix in $\Gamma\cap O_g$ that is an odd permutation of the elements of g will either be h or have the angle $\frac{2\pi}{3}$ between itself and h.

Thus the orbit of $g=C(\theta)$, $0<\theta<\frac{\pi}{3}$, O_g, intersects Γ in exactly one point in each $\{C(t)\mid \frac{k\pi}{3}<t<\frac{(k+1)\pi}{3}\}$ for $0\le k\le 5$. So, make the identification of each point in $\Gamma\cap O_g$ with $g=C(\theta)$ for $0\le\theta\le\frac{\pi}{3}$. In other words, for any $0\le\theta\le\frac{\pi}{3}$, identify or 'glue' together the points $C(\theta),C(\frac{2\pi}{3}-\theta),C(\frac{2\pi}{3}+\theta),C(\frac{4\pi}{3}-\theta),C(\frac{4\pi}{3}+\theta),C(2\pi-\theta)$.

Intuitively, imagine a pizza with six slices of equal area, and taking one slice and folding it onto the counterclockwise adjacent slice, and repeating this 4 more times.

We then end up with a 'pizza sandwich' with our identification being on the crust or 'edge'.

With this in mind, the pre-image of any open subset of $\{C(\theta) \mid 0 \leq \theta \leq \frac{\pi}{3}\}$ will be open under our identification. (The preimage of the open half-interval at either of the endpoints 'unfolds' to become open intervals about $\Gamma \cap O_D$ where O_D is a Veronese surface.) Thus, our identification is continuous.

Thus $S^4/SO(3)$ can be viewed as $\{C(\theta) \mid 0 \leq \theta \leq \frac{\pi}{3}\}$, as for each $0 \leq \theta \leq \frac{\pi}{3}$ corresponds to a unique orbit $C(\theta)$ under our identification due to the statements above. And as Γ intersects every $SO(3)$ orbit, we have that the set $\{C(\theta) \mid 0 \leq \theta \leq \frac{\pi}{3}\}$ sees every orbit of $SO(3)$ acting on S^4.

The explicit identification written three paragraphs earlier shows that for $\theta = 0, \frac{\pi}{3}$, $C(\theta)$ corresponds to each Veronese surface respectively. Thus $S^4/SO(3)$ is topologically a closed interval whose endpoints correspond to the two Veronese surfaces. □

e). Parameterize Γ by S^1: find a unit speed curve $C : S^1 \to \Gamma$, $\theta \mapsto C(\theta)$, $0 \leq \theta \leq 2\pi$, $C(0) = C(2\pi)$ which sweeps out Γ.

Proof. Using the orthonormal basis constructed in problem 2a, we can write $C(\theta) = \cos(\theta)e_1 + \sin(\theta)e_2$. And $C(0) = e_1 = C(2\pi)$, and

$$\begin{aligned} \|C'(\theta)\|^2 &= \langle C'(\theta), C'(\theta)\rangle \\ &= (-\sin(\theta))^2\|e_1\|^2 - 2\sin(\theta)\cos(\theta)\langle e_1, e_2\rangle + (\cos(\theta))^2\|e_2\|^2 = 1 \end{aligned}$$

Thus $C(\theta)$ has unit speed. □

f). Consider the map $\Psi : SO(3) \times S^1 \to S^4$ given by $\Psi(g, \theta) \mapsto g \cdot C(\theta)$. Show that the pullback of the standard metric ds^2 on S^4 (the one induced from the inner product on $\mathbb{V}$) can be expressed as

$$\Psi^* ds^2 = d\theta^2 + a(\theta)^2\sigma_1^2 + b(\theta)^2\sigma_2^2 + c(\theta)^2\sigma_3^2$$

where the σ_i form the standard basis for the space of left–invariant one–forms on $SO(3)$.

Proof. For a fixed $x_0 \in \Gamma$, there exists $\theta_0 \in [0, 2\pi]$ such that $x_0 = C(\theta_0)$. Define $\Psi^{x_0} : SO(3) \to S^4$ by $\Psi^{x_0}(g) = g \cdot x_0$. For a fixed $h \in SO(3)$ define $F^h : S^1 \to S^4$ by $F^h(\theta) = h \cdot C(\theta)$. A previous result gives

$$T_{(g,\theta)}(SO(3) \times S^1) \cong T_g SO(3) \oplus T_\theta S^1.$$

Thus we can view the differential $d\Psi_{(g,\theta)}$ as $(d\Psi^{x_0})_g \times (dF^g)_\theta$. Thus for $(g,\theta_1),(h,\theta_2) \in T_{(g,\theta)}(SO(3)\times S^1)$,

$$\begin{aligned}\Psi^* ds^2((g,\theta_1),(h,\theta_2)) &= ds^2(d\Psi(g,\theta_1), d\Psi(h,\theta_2)) = ds^2((d\Psi^{x_0}(g), dF^g(\theta_1)),(d\Psi^{x_0}(h), dF^g(\theta_2)))\\ &= ds^2((d\Psi^{x_0}(g),0),(d\Psi^{x_0}(h),0) + ds^2(0, dF^g(\theta_1)),(0, dF^g(\theta_2)))\end{aligned}$$

For a fixed θ, $dF^g(\theta)=0$. Hence

$$\Psi^* ds^2((g,\theta),(h,\theta)) = ds^2((d\Psi^{x_0}(g),0),(d\Psi^{x_0}(h),0).$$

As $T_gSO(3)$ is equivalent to the space of skew symmetric matrices for any $g\in SO(3)$, and that

$$E_1 = \begin{bmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix},$$

is an orthogonal basis for the space of skew symmetric matrices, we have that

$$ds^2((d\Psi^{x_0}(g),0),(d\Psi^{x_0}(h),0) = a(\theta)^2\sigma_1^2 + b(\theta)^2\sigma_2^2 + c(\theta)\sigma_3^2,$$

for $\sigma_1,\sigma_2,\sigma_3$ the dual basis associated to E_1,E_2,E_3.

□

g). Compute the functions $a(\theta), b(\theta), c(\theta)$.

h). Compute the three–dimensional volume of the $SO(3)$–orbit through $C(\theta)\in\Gamma$. The result will be a function of θ.