1. Let E be a vector bundle over M and $p \neq q$ be two points of M.

   By definition of vector bundle, $\exists$ local trivialization $\pi^{-1}(U) \overset{\varphi}{\to} U \times \mathbb{R}^k$ with $q \in U$. Let $h : M \to \mathbb{R}$ be a smooth bump function with $h(p) = 0$ and $h(q) = 1$ and $\text{supp}(h) \subset U$.

   For $x \in U$, $\pi^{-1}(x) \overset{\varphi}{\cong} x \times \mathbb{R}^k$ (isomorphic vector spaces). Now take a basis $e_1, ..., e_k$ of $\mathbb{R}^k$, so for each $x \in U$, we have that $\varphi^{-1}(x, e_i)$ is a basis for $\pi^{-1}(x)$.

   Define $s : M \to E$ by $x \mapsto \begin{cases} \varphi^{-1}(x, h(x)e_1) & x \in U \\ 0_x & x \notin U \end{cases}$. Where 0_x means the 0 vector in the vector space $\pi^{-1}(x)$. Also s is smooth as it is the product of smooth functions $h(-) \cdot \varphi^{-1}(-, e_1)$.

   Then for $x \notin U$, $\pi(s(x)) = \pi(0_x) = x$, and for $x \in U$,we have $\pi(s(x)) = \pi(\varphi^{-1}(x, h(x)e_1)) = \pi_1(x, h(x)e_1) = x$, so s is a section. (using comm. diagram for local trivializations pg. 104 of Lee). And we have $s(q) = \varphi^{-1}(q, e_1) \neq 0$, while $s(p) = 0$ if $p \notin U$ and $s(p) = \varphi^{-1}(p, 0) = 0$ if $p \in U$.

2. Let $E_{\mathbb{C}}$ be a complex line bundle and $E_{\mathbb{R}}$ be the same line bundle regarded as a real vector bundle (so $E_{\mathbb{C}} = E_{\mathbb{R}}$ and both use the same $\pi : E \to M$, they have different local trivializations).

   Suppose $E_{\mathbb{C}}$ is trivial, then $\exists$ global trivialization $\phi_{\mathbb{C}} : E_{\mathbb{C}} \to M \times \mathbb{C}$.
   Let $\phi_{\mathbb{R}}$ be the composition $E_{\mathbb{R}} \overset{\phi_{\mathbb{C}}}{\to} M \times \mathbb{C} \overset{\text{id}\times f}{\to} M \times \mathbb{R}^2$, where $\mathbb{C} \ni z \overset{f}{\mapsto} (\Re z, \Im z)$. Since $\phi_{\mathbb{C}}$ is a trivialization, $\pi_1 \circ \phi_{\mathbb{C}} = \pi$. Then $\pi_1 \phi_{\mathbb{R}}(v) = \pi_1(\pi_1 \phi_{\mathbb{C}}(v), \Re \pi_2(\phi_{\mathbb{C}}(v)), \Im \pi_2(\phi_{\mathbb{C}}(v))) = \pi_1 \phi_{\mathbb{C}}(v) = \pi(v)$, so the pg. 104 diagram commutes for $\phi_{\mathbb{R}}$ ($\pi_1 \circ \phi_{\mathbb{R}} = \pi$).
   Also $\pi^{-1}(x) \overset{\phi_{\mathbb{C}}}{\cong} x \times \mathbb{C} \overset{\text{id}\times f}{\cong} x \times \mathbb{R}^2$ are $\mathbb{R}$-vector space isomorphisms, and so their composition $\phi_{\mathbb{R}}$ is too, hence $\phi_{\mathbb{R}}$ is a global trivialization for $E_{\mathbb{R}}$.

   Conversely...

3. Let E be a real vector bundle over compact manifold M s.t. $\text{rk } E > \dim M$. Show E has a non-vanishing section.

   Recall the transversality theorem says: Let A, B be smooth manifolds, $f : A \to B$ be smooth, and $C \subset B$ a regular submanifold. If $f \pitchfork C$ then $f^{-1}(C)$ is a regular submanifold of A. Moreover smooth functions transverse to C are dense in C^k norm.

   Also note that when $\dim A + \dim C < \dim B$, we have $f \pitchfork C \iff f(B) \cap C = \emptyset$.

   For the zero section $s : M \to E$ and the submanifold $Z = \{0_x : x \in M\} \subset E$, we have by the denseness part of transversality theorem that there is a smooth function $f : M \to E$ that is arbitrarily C^1 close to s, and $f \pitchfork Z$. By assumption $\dim M + \dim Z = \dim M < \text{rk} E \leq \dim E$, so $f \pitchfork Z$ implies $f(M) \cap Z = \emptyset$ i.e. $f(x) \neq 0 \forall x \in M$.

   Since f is arbitrarily C^1 close to s, $\pi \circ f$ is arbitrarily close to $\pi \circ s = \text{id}$.
   Then since identity is a diffeomorphism so is $\pi \circ f$ (I haven't done details of this yet).
   Now we take $\sigma = f \circ (\pi \circ f)^{-1} : M \to E$, and have that $\pi \circ \sigma = \pi \circ f \circ (\pi \circ f)^{-1} = \text{id}$, so σ is a section. And for $x \in M$, set $y = (\pi \circ f)^{-1}(x) \in M$ then $\sigma(x) = f(y) \neq 0$ so σ is a non-vanishing section.

4. Let $\pi : E \to M$ be a smooth vector bundle of rank k with $\{U_\alpha\}_{\alpha\in A}$ an open cover of M having local trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$. For $U_\alpha \cap U_\beta \neq \emptyset$, let $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(\mathbb{R})$, be defined by $\phi_\alpha\phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$.

   For $\alpha, \beta, \gamma \in A$ with $V := U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. For $p \in V$ and $v \in \mathbb{R}^k$, we have: $(p, v) \overset{\phi_\beta\phi_\gamma^{-1}}{\mapsto} (p, \tau_{\beta\gamma}(p)v) \overset{\phi_\alpha\phi_\beta^{-1}}{\mapsto} (p, \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)v)$, while the composition is $\phi_\alpha\phi_\gamma^{-1}$ which is defined by $(p, v) \mapsto (p, \tau_{\alpha\gamma}(p)v)$. Hence $\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)v = \tau_{\alpha\gamma}(p)v$ and since $v \in \mathbb{R}^k$ was arbitrary we have $\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p)$.

5. (5-2) In example2 of pg. 105 on Lee, show $E = [0, 1] \times \mathbb{R}/\{(0, y) \sim (1, -y)\}$ with $\pi(x, y) = x$ is a vector bundle of rank 1 over S^1 (thought of as $\mathbb{R}/\mathbb{Z}$ via $x \leftrightarrow e^{2\pi ix}$ for now), and is non-trivial.

   We have the local trivializations:
   For $U = S^1 \setminus \{1\} = \mathbb{R}/\mathbb{Z} \setminus \{1\}$, $\phi_1 : \pi^{-1}(U) \to U \times \mathbb{R}$ by $(x, y) \mapsto (x, y)$ and
   $V = S^1\setminus\{-1\} = \mathbb{R}/\mathbb{Z}\setminus\{\frac{1}{2}\}$, $\phi_2 : \pi^{-1}(V) \to V \times \mathbb{R}$ by $(x, y) \mapsto \begin{cases} (x, y) & 0 \leq x < \frac{1}{2} \\ (x-1, -y) & \frac{1}{2} < x \leq 1 \end{cases}$.

   I'll verify that ϕ_2 is a local trivialization (likewise ϕ_1 is too).

   First $\phi_2(1, -y) = (0, y) = \phi_2(0, y)$ so ϕ_2 is well-defined. And to see the pg. 104 diagram commutes, we have for $0 \leq x < \frac{1}{2}$ and $y \in \mathbb{R}$ that $\pi(x, y) = x = \pi_1(\phi_2(x, y))$. Similarly for $\frac{1}{2} < x \leq 1$ we get $\pi_1\phi_2(x, y) = x - 1 = x = \pi(x, y)$ (since $x - 1 = x$ in $\mathbb{R}/\mathbb{Z} = S^1$).

   *Also restricting ϕ_2 to a fiber E_x gives the vector space isomorphism of $\mathbb{R}$: $y \mapsto y$ or $y \mapsto -y$ depending on if $0 \leq x < \frac{1}{2}$ or $\frac{1}{2} < x \leq 1$ resp.

   If E were trivial we would have $E \cong S^1 \times \mathbb{R}$ in other words the Mobius strip is diffeomorphic to the cylinder. But the boundary of the Mobius strip is connected while the boundary of the cylinder is disconnected, contradiction, hence E is not a trivial vector bundle.

   (5-6): We saw in * of problem 5-2 that the $\tau : U \cap V \to GL_1(\mathbb{R})$ was given by $\tau(x) = \begin{cases} (1) & 0 < x < \frac{1}{2} \\ (-1) & \frac{1}{2} < x < 1 \end{cases}$, which translates to $\tau(z) = \begin{cases} (1) & \Im z > 0 \\ (-1) & \Im z < 0 \end{cases}$ under $x \mapsto e^{2\pi ix}$. Since the bundle F from problem 5-4 is unique upto vector bundle isomorphism, $F \cong E$ as they both have the same transition functions τ on the same set $U \cap V$.

   (5-12): The tautological bundle over $G_{\mathbb{R}}(1, 2) = \mathbb{R}P^1$ is $E = \{([x], y) : [x] \in \mathbb{R}P^1, y \in [x]\} \subset \mathbb{R}P^1 \times \mathbb{R}^2$ with $\pi([x], y) = [x]$. Let $E' \overset{\pi}{\to} S^1$ be the mobius bundle of 5-2.

   For $[x_1, x_2] \in \mathbb{R}P^1$ and $x_2 \neq 0$, let $\theta \in (0, \pi)$ be the angle the line makes measured clockwise from x-axis ($\theta := 0$ for $x_2 = 0$).
   For $y \in \text{span}_{\mathbb{R}}(x_1, x_2)$ let $\lambda_y \in \mathbb{R}$ s.t. $y = \lambda_y\vec{u}$ where $|\vec{u}| = 1$ and $\vec{u}$ is on the line $[x_1, x_2]$ with positive y coordinate. $(*)$ Note for $c \in \mathbb{R}$ and $z \in \text{span}_{\mathbb{R}}(x_1, x_2)$, we have $cy = (c\lambda_y)\vec{u}$ and $y + z = (\lambda_y + \lambda_z)\vec{u} \Rightarrow \lambda_{y+z} = \lambda_y + \lambda_z$ and $\lambda_{cy} = c\lambda_y$.

   Define $F : E \to E'$ by $F([x_1, x_2], y) = \begin{cases} (\frac{\theta}{\pi}, \lambda_y) & x_2 \neq 0 \\ (0, x_1) \sim (1, -x_1) & x_2 = 0 \end{cases}$.

Now to make the diagram $\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \pi\downarrow & & \downarrow\pi' \\ \mathbb{R}P^1 & \xrightarrow{f} & S^1 \end{array}$ commute, we set $f([x_1, x_2]) := e^{2i\theta}$. Finally let $p = [x_1, x_2] \in \mathbb{R}P^1$. To check $F|_{E_p} : E_p \to E'_{f(p)}$ is linear we'll use $(*)$ above. For $(p, y), (p, z) \in E_p, c \in \mathbb{R}$, we have:
$F|_{E_p}(p, y + cz) = (\frac{\theta}{\pi}, \lambda_{y+cz}) \overset{*}{=} (\frac{\theta}{\pi}, \lambda_y) + c(\frac{\theta}{\pi}, \lambda_z) = F|_{E_p}(p, y) + cF|_{E_p}(p, z)$.
Similarly, we can verify that $G : E' \to E$ by $(x, y) \mapsto ([\cos \pi x, \sin \pi x], (y \cos \pi x, y \sin \pi x))$ is a bundle inverse to F, so F is a vector bundle isomorphism. Now since E' is non-trivial by 5-2, so is E.

(For 5-2,5-6,5-12 I don't know if the maps are diffeomorphisms).

6. Let E be the tautological line bundle over $\mathbb{R}P^n = G_{\mathbb{R}}(1, n+1)$. Then $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ by $[x_1, x_2] \mapsto [x_1, x_2, 0, ..., 0]$. Now following the hint, suppose there is a global trivialization $\phi : E \to \mathbb{R}P^n \times \mathbb{R}$. Then set $E' = \{([x_1, x_2, 0, ..., 0], (y_1, y_2, 0, ..., 0)) : (y_1, y_2) \in \text{span}_{\mathbb{R}}(x_1, x_2)\} = \pi^{-1}(\mathbb{R}P^1)$. So E' is the same as the tautological bundle over $\mathbb{R}P^1$, and taking $\phi|_{E'} : E' \to \pi(E') \times \mathbb{R} = \mathbb{R}P^1 \times \mathbb{R}$ gives a trivialization of the tautological line bundle over $\mathbb{R}P^1$ contradicting 5-12. Hence the tautological line bundle over $\mathbb{R}P^n$ is also non-trivial.

   The unit sphere bundle $SE := \amalg_{p \in M}\{x \in E_p : |x| = 1\} = \{([x], \pm\frac{x}{|x|}) : [x] \in \mathbb{R}P^n\}$. $SE \cong S^n$ by $([x], u) \mapsto u \in S^n$. With this identification, the projection $SE \to \mathbb{R}P^1$ corresponds to the map $S^n \to \mathbb{R}P^1$ by $u \mapsto [u]$ (thinknig of $\mathbb{R}P^n$ as $S^n/\{x \sim -x\}$ it corresponds to standard projection $S^n \to S^n/\{x \sim -x\}$).

7. Let E be tautological line bundle over $\mathbb{C}P^n = Gr_{\mathbb{C}}(1, n+1)$.

   (a) By definition $SE = \{([z_0, ..., z_n], u) : [z_0, ..., z_n] \in \mathbb{C}P^n, u = e^{i\theta}\frac{\vec{z}}{|\vec{z}|}, 0 \leq \theta < 2\pi\}$ where $\vec{z} = (z_0, ..., z_n)$ (so $|u| = 1$).
   Then we have the map $SE \to S^{2n+1}$ by $([z_0, ..., z_n], u) \mapsto u$ where we view S^{2n+1} as sitting inside $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$.

   (b) For $n = 1$ the Hopf map is defined as $H : S^3 \to S^2$ by $(z_0, z_1) \mapsto (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$. We're thinking of $S^3 \subset \mathbb{C}^2$ and $S^2 \subset \mathbb{C} \times \mathbb{R}$.

   We have the maps $\mathbb{C}P \setminus \{[1, 0]\} \to \mathbb{C} \to S^2 \setminus \{0, 0, 1\}$ by $[z_0, z_1] \mapsto \frac{z_0}{z_1} = x + iy \mapsto (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2})$ (here $S^2 \subset \mathbb{R}^3$, last map is inverse of stereoproj).
   Using $x + iy = \frac{z_0}{z_1}$, we get:
   $\frac{2x}{1+x^2+y^2} = \frac{\frac{z_0}{z_1} + \frac{\bar{z}_0}{\bar{z}_1}}{1 + \frac{|z_0|^2}{|z_1|^2}} = \frac{z_0\bar{z}_1 + \bar{z}_0 z_1}{|z_1|^2 + |z_0|^2} = \frac{2\Re(z_0\bar{z}_1)}{|z_0|^2 + |z_1|^2}$.
   Likewise $\frac{2y}{1+x^2+y^2} = \frac{2\Im(z_0\bar{z}_1)}{|z_1|^2 + |z_2|^2}$ and $\frac{x^2+y^2-1}{1+x^2+y^2} = \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2}$.

   So we are led to define $P : \mathbb{C}P \to S^2 \subset \mathbb{C} \times \mathbb{R}$ by $(z_0, z_1) \mapsto (\frac{2z_0\bar{z}_1}{|z_0|^2 + |z_1|^2}, \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2})$. And P ends up being a diffeomorphism of $\mathbb{C}P$ with S^2.

Now to show that $SE \xrightarrow{\pi} \mathbb{C}P$ is the Hopf map, we'll show the diagram:

$$\begin{array}{ccc} SE & \xrightarrow{\sim} & S^3 \\ \pi\downarrow & & {\scriptstyle H}\downarrow \\ \mathbb{C}P & \xrightarrow{P} & S^2 \end{array}$$

commutes where the $\sim$ is the diffeo from part (a). Let $(z_0, z_1) \in S^3$ (so $|z_0|^2 + |z_1|^2 = 1$), then:

$$(z_0, z_1) \overset{\sim}{\mapsto} ([z_0, z_1], (z_0, z_1)) \overset{\pi}{\mapsto} [z_0, z_1] \overset{P}{\mapsto} (2z_0\overline{z}_1, |z_0|^2 - |z_1|^2) = H(z_0, z_1).$$

For $n > 1$ the Hopf map is defined as the restriction of the natural projection $\rho : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}P^n$ to $S^{2n+1} \subset \mathbb{C}^{n+1}$. So we want to show the diagram:

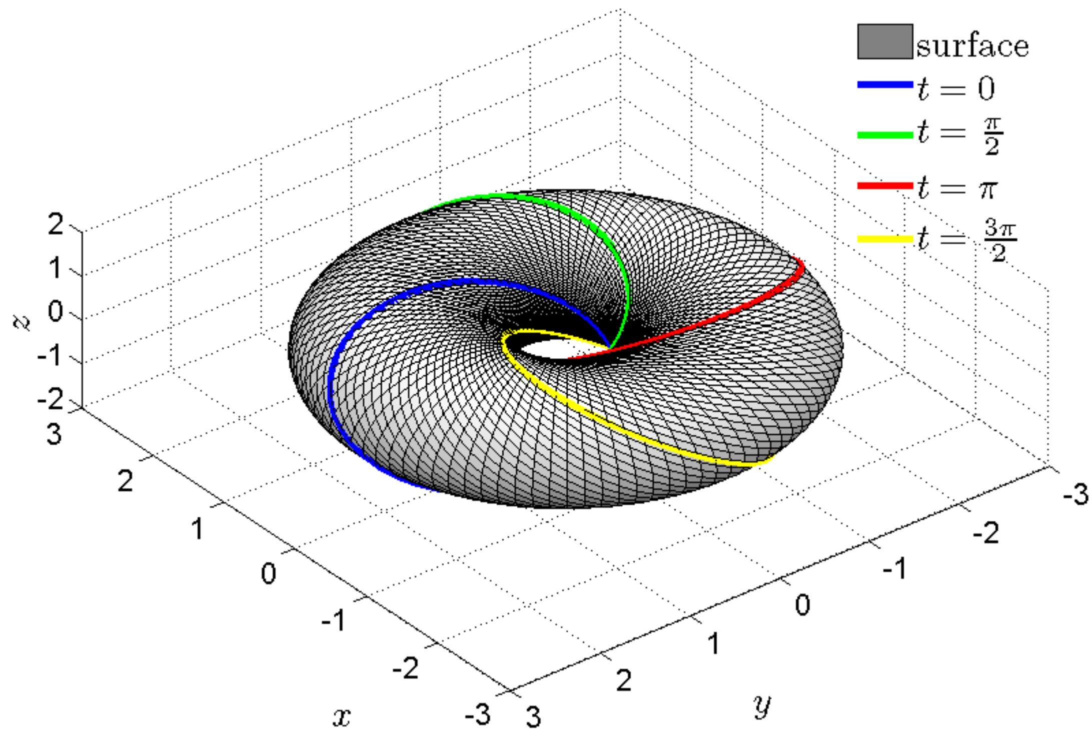
$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{\sim} & SE \\ {\scriptstyle H}\downarrow & \swarrow & \\ \mathbb{C}P^n & & \end{array}$$

commutes (where the diagonal arrow is π).
For $\vec{z} \in S^{2n+1}$ we have $\vec{z} \overset{\sim}{\mapsto} ([\vec{z}], \vec{z}) \overset{\pi}{\mapsto} [\vec{z}] = H(\vec{z})$ so it does commute.

(c) For this part we'll think of $S^3 \subset \mathbb{R}^4$. We have $S^3 \setminus (0,0,0,1) \xrightarrow{f} \mathbb{R}^3$ by $(x, y, z, w) \mapsto (\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w})$. Now we want to try to sketch $f(\pi^{-1}([z_0, z_1]))$ for various $[z_0, z_1]$ while identifying $\mathbb{C}^2$ with $\mathbb{R}^4$. Setting $\vec{u} = \frac{(z_0,z_1)}{|(z_0,z_1)|} = (x + iy, z + iw)$ we get:
$\pi^{-1}([z_0, z_1]) = \{e^{i\theta}\vec{u} : 0 \le \theta < 2\pi\} = \{(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z\cos\theta - w\sin\theta, z\sin\theta + w\cos\theta) : 0 \le \theta < 2\pi\}$. So the fiber over $[z_0, z_1]$ is:

$$\{(\frac{x\cos\theta - y\sin\theta}{1 - z\sin\theta - w\cos\theta}, \frac{x\sin\theta + y\cos\theta}{1 - z\sin\theta - w\cos\theta}, \frac{z\cos\theta - w\sin\theta}{1 - z\sin\theta - w\cos\theta}) : 0 \le \theta < 2\pi\}$$

Let's always write $[z_0, z_1]$ with a unit length representative. Fixing z_0, we get from $|z_0|^2 + |z_1|^2 = 1$ that the magnitude of z_1 is fixed (but the argument may still vary) i.e. $z_1 = r_1 e^{it}$ for $0 \le t \le 2\pi$. Now the fiber over $[z_0, z_1]$ can be described as $\{e^{i\theta}(z_0, r_1 e^{it}) : 0 \le \theta, t < 2\pi\}$ (before going to $\mathbb{R}^3$) which seems like a torus. For an example in $\mathbb{R}^3$ take $z_0 = \frac{1}{\sqrt{2}} \Rightarrow z_1 = \frac{1}{\sqrt{2}}e^{it}$ to get the surface:
$\{(\frac{\cos\theta}{\sqrt{2} - (\cos t\sin\theta + \sin t\cos\theta)}, \frac{\sin\theta}{\sqrt{2} - (\cos t\sin\theta + \sin t\cos\theta)}, \frac{\cos t\cos\theta - \sin t\sin\theta}{\sqrt{2} - (\cos t\sin\theta + \sin t\cos\theta)}) : 0 \le t, \theta < 2\pi\}$. With a little help from our matlab using friends we see it is in fact a torus. Also we can fix some values of t to see fibers of indivudaul $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{it}]$ along our torus (see attached picture).


surface
$t = 0$
$t = \frac{\pi}{2}$
$t = \pi$
$t = \frac{3\pi}{2}$
z
2
1
0
-1
-2
x
3
2
1
0
-1
-2
-3
y
3
2
1
0
-1
-2
-3