P Sj, 2.23.i. One of the greatest advances in theoretical physics of the nineteenth century was Maxwell's formulation of the equations of electromagnetism

$$\begin{aligned} \nabla \times E &= -\frac{1}{c}\frac{\partial B}{\partial t} && \text{(Faraday's law)} \\ \nabla \times H &= \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial D}{\partial t} && \text{(Ampere's law)} \\ \nabla \cdot D &= 4\pi\rho && \text{(Gauss' law)} \\ \nabla \cdot B &= 0 && \text{(no magnetic monopoles).} \end{aligned}$$

Here:

$$\begin{aligned} c &: \text{speed of light} \\ E &: \text{electric field} \\ H &: \text{magnetic field} \\ J &: \text{current density} \\ \rho &: \text{charge density} \\ B &: \text{magnetic induction} \\ D &: \text{dielectric displacement} \end{aligned}$$

E, H, J, B, D are vector fields and ρ is a function on $\mathbb{R}^3$. All depend on time t.

In space-time $\mathbb{R}^4$, with coordinates (x_1, x_2, x_3, x_4), where $x_4 := ct$, we introduce forms

$$\begin{aligned} \alpha &= (E_1 dx_1 + E_2 dx_2 + E_3 dx_3)dx_4 + B_1 dx_2 dx_3 + B_2 dx_3 dx_1 + B_3 dx_1 dx_2, \\ \beta &= -(H_1 dx_1 + H_2 dx_2 + H_3 dx_3)dx_4 + D_1 dx_2 dx_3 + D_2 dx_3 dx_1 + D_3 dx_1 dx_2, \\ \gamma &= \frac{1}{c}(J_1 dx_2 dx_3 + J_2 dx_3 dx_1 + J_3 dx_1 dx_2)dx_4 - \rho dx_1 dx_2 dx_3. \end{aligned}$$

Show Maxwell's equations are equivalent to

$$\begin{aligned} d\alpha &= 0, \\ d\beta + 4\pi\gamma &= 0. \end{aligned}$$

A: The differential in $\mathbb{R}^4$ is pretty straightforward. For a basic k-form $f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k(\mathbb{R}^4)$, the differential is a $(k+1)$-form defined to be:

$$d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{i=1}^{4} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

ie. take the partials with each coordinate, and tack the corresponding $dx_i \wedge -$ to the wedges.

We can see that each α, β have electric and magnetic pieces:

$$\alpha = \underbrace{(E_1 dx_1 + E_2 dx_2 + E_3 dx_3)dx_4}_{\text{electric}} + \underbrace{B_1 dx_2 dx_3 + B_2 dx_3 dx_1 + B_3 dx_1 dx_2}_{\text{magnetic}},$$

$$\beta = \underbrace{-(H_1 dx_1 + H_2 dx_2 + H_3 dx_3)dx_4}_{\text{magnetic}} + \underbrace{D_1 dx_2 dx_3 + D_2 dx_3 dx_1 + D_3 dx_1 dx_2}_{\text{electric}}.$$

Let's start small. Since the differential is linear, we can look at summands of α, β, γ individually and generalize. The first summand of α is $E_1 dx_1 \wedge dx_4$. Then the differential

$$\begin{aligned} d(E_1 dx_1 \wedge dx_4) &= \sum_{i=1}^{4} \frac{\partial E_1}{\partial x_i} dx_i \wedge dx_1 \wedge dx_4 \\ &= \frac{\partial E_1}{\partial x_2} dx_2 \wedge dx_1 \wedge dx_4 + \frac{\partial E_1}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_4 \\ &= -\frac{\partial E_1}{\partial x_2} dx_1 \wedge dx_2 \wedge dx_4 - \frac{\partial E_1}{\partial x_3} dx_1 \wedge dx_3 \wedge dx_4, \end{aligned}$$

after getting rid of terms with repeated indices and rearranging them in increasing order. We can infer the other two (they look similar, with different permutations of $i = 1, 2, 3$). That's the electric field part of α.

The first summand of the magnetic part of α is $B_1 dx_2 \wedge dx_3$. This is straightforward by now:

$$\begin{aligned} d(B_1 dx_2 \wedge dx_3) &= \sum_{i=1}^{4} \frac{\partial B_1}{\partial x_i} dx_i \wedge dx_2 \wedge dx_3 \\ &= \frac{\partial B_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial B_1}{\partial x_4} dx_2 \wedge dx_3 \wedge dx_4. \end{aligned}$$

The other two terms will also be permutations of the indices of the above.

Putting them all together, we emerge with

$$\begin{aligned} d\alpha_{\text{electric}} = \left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) dx_1 \wedge dx_2 \wedge dx_4 + \left(\frac{\partial E_3}{\partial x_1} - \frac{\partial E_3}{\partial x_1}\right) dx_3 \wedge dx_1 \wedge dx_4 \\ + \left(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}\right) dx_2 \wedge dx_3 \wedge dx_4. \end{aligned}$$

The curl(E) term is palpable here. We can see each component of the curl in each expression above. We can rewrite the above more explicitly as $\sum_{i=1}^{3} (\nabla \times E)_i \hat{dx_i}$, where $(\nabla \times E)_i$ is the i^{th} component of the curl, and $\hat{dx_i} := dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_4$.

Calculating the magnetic part:

$$\begin{aligned} d\alpha_{\text{magnetic}} = &\left(\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3 \\ &+ \frac{\partial B_1}{\partial x_4} dx_2 \wedge dx_3 \wedge dx_4 \\ &+ \frac{\partial B_2}{\partial x_4} dx_3 \wedge dx_1 \wedge dx_4 \\ &+ \frac{\partial B_3}{\partial x_4} dx_1 \wedge dx_2 \wedge dx_4. \end{aligned}$$

The first term above we recognize as $(\nabla \cdot B)\, dx_1 \wedge dx_2 \wedge dx_3$. Since this is the only $dx_1 \wedge dx_2 \wedge dx_3$ term, linearly independent with all the others, saying that $d\alpha = 0$ implies that $\nabla \cdot B = 0$, ie. no magnetic monopoles.

Since $x_4 = ct \implies \frac{\partial}{\partial x_4} = \frac{dx_4}{dt}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}$, we can rewrite the magnetic part as a sum: $\sum_{i=1}^{3} \frac{1}{c}\frac{\partial B_i}{\partial t} d\hat{x}_i$.

Altogether, we have

$$d\alpha = \sum_{i=1}^{3} \left((\nabla \times E)_i + \frac{1}{c}\frac{\partial B_i}{\partial t} \right) d\hat{x}_i = 0,$$

Since $d\hat{x}_i$ area all linearly independent, this means each coefficient is 0, ie.

$$(\nabla \times E)_i = \frac{-1}{c}\frac{\partial B_i}{\partial t},$$

for $i = 1, 2, 3$. This is just Faraday's law in each component.

The second is a similar computation; we'll skip it. □