U_i being open, U_j' is dense in U_i and hence in V. Let

$$U_{ij}'=\left\{x\in U_j' \,|\, T_xf\left(\frac{\partial}{\partial x^i}\right)_x,\ T_xf\left(\frac{\partial}{\partial x^j}\right)_x \text{ linearly independent}\right\};$$

we show that U_{ij}' is dense in V. If this were not the case there is an open set $W\subset V$ such that

$$\lambda(x)T_xf\left(\frac{\partial}{\partial x^i}\right)_x+T_xf\left(\frac{\partial}{\partial x^j}\right)_x=0$$

for a certain smooth function λ nonzero on W. Let $c\colon(-\varepsilon,\varepsilon)\to W$ be an integral curve of the vector field $\lambda(\partial/\partial x^i)-(\partial/\partial x^j)$. Then $(f\circ c)'(t)=T_{c(t)}f(c'(t))=0$ so that $f\circ c$ is constant on $(-\varepsilon,\varepsilon)$ contradicting injectivity of f.)

2.3 EXTERIOR ALGEBRA

The calculus of Cartan concerns exterior differential forms, which are sections of a vector bundle of linear exterior forms on the tangent spaces of a manifold. We begin with the exterior algebra of a vector space and extend this fiberwise to a vector bundle. As with tensor fields, the most important case is the tangent bundle of a manifold, which is considered in the next section.

2.3.1 Definition. *Let E be a finite-dimensional real vector space. Let $\Omega^0(E)=\boldsymbol{R}$. $\Omega^1(E)=E^*$, and, in general, $\Omega^k(E)=L_a^k(E,\boldsymbol{R})$, the vector space of skew symmetric k multilinear maps or* ***exterior k-forms*** *on E.*

We leave as an easy exercise the fact that $\Omega^k(E)$ is a vector subspace of $T_k^0(E)$.

Recall that the permutation group on k elements, denoted S_k, consists of all bijections $\varphi\colon\{1,\ldots,k\}\to\{1,\ldots,k\}$ together with the structure of a group under composition. Clearly, S_k has order $k!$. Letting $(\tilde{\boldsymbol{R}},\times)$ denote $\boldsymbol{R}\backslash\{0\}$ with the multiplicative group structure, we have a homomorphism *sign*: $S_k\to(\tilde{\boldsymbol{R}},\times)$. That is, for $\sigma,\tau\in S_k$, $\mathrm{sign}(\sigma\circ\tau)=(\mathrm{sign}\,\sigma)(\mathrm{sign}\,\tau)$. The image of *sign* is the subgroup $\{-1,1\}$, while its kernel consists of the subgroup of even permutations. One other fact we shall need is the following, which the reader can easily check: If G is a group and $g_0\in G$, the map $R_{g_0}\colon G\to G\colon g\mapsto gg_0$ is a bijection.

2.3.2 Definition. ***The alternation mapping*** *$A\colon T_k^0(E)\to T_k^0(E)$ (as before, we do not index the A) is defined by*

$$At(e_1,\ldots,e_k)=\frac{1}{k!}\sum_{\sigma\in S_k}(\mathrm{sign}\,\sigma)t(e_{\sigma(1)},\ldots,e_{\sigma(k)})$$

where the sum is over all $k!$ elements of S_k.

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2.3.3 Proposition. *A is a linear mapping onto $\Omega^k(E)$, $A|\Omega^k(E)$ is the identity, and $A \circ A = A$.*

Proof. Linearity of A follows at once. If $t \in \Omega^k(E)$, then

$$\begin{aligned} At(e_1,\ldots,e_k) &= \frac{1}{k!}\sum_{\sigma\in S_k}(sign\,\sigma)t(e_{\sigma(1)},\ldots,e_{\sigma(k)}) \\ &= \frac{1}{k!}\sum_{\sigma\in S_k} t(e_1,\ldots,e_k) \\ &= t(e_1,\ldots,e_k) \end{aligned}$$

since $(sign\,\sigma)^2 = 1$ and S_k has order $k!$. Second, for $t \in T_k^0(E)$ we have

$$\begin{aligned} At(e_1,\ldots,e_k) &= \frac{1}{k!}\sum_{\sigma\in S_k}(sign\,\sigma)t(e_{\sigma(1)},\ldots,e_{\sigma(k)}) \\ &= \frac{1}{k!}\sum_{\sigma\in S_k}(sign\,\sigma\tau)t(e_{\sigma\tau(1)},\ldots,e_{\sigma\tau(k)}) \\ &= (sign\,\tau)At(e_{\tau(1)},\ldots,e_{\tau(k)}) \end{aligned}$$

since $\sigma \mapsto \sigma\tau$ is a bijection and *sign* is a homomorphism. This proves the first two assertions, and the last follows from them. ■

Then we may define the *exterior product* as follows.

2.3.4 Definition. *If $\alpha \in T_k^0(E)$ and $\beta \in T_l^0(E)$, define $\alpha\wedge\beta \in \Omega^{k+l}(E)$ by $\alpha\wedge\beta = (k+l)!/k!l!\ A(\alpha\otimes\beta)$. (Again, we do not index $\wedge$.) In particular, for $\alpha \in T_0^0(E) = \mathbb{R}$, we put $\alpha\wedge\beta = \beta\wedge\alpha = \alpha\beta$.*

There are several possible conventions for defining the wedge product $\wedge$. The one here conforms to Spivak [1965], and Bourbaki [1971] but not to Kobayashi–Nomizu [1963] or Guillemin–Pollack [1976]. See Robbin [1974] for a lively discussion of what conventions are possible.

Our definition of $\alpha\wedge\beta$ is the one that eliminates the largest number of constants later. The reader should prove that, for exterior forms,

$$(\alpha\wedge\beta)(e_1,\ldots,e_{k+l}) = \Sigma'(sign\,\sigma)\alpha(e_{\sigma(1)},\ldots,e_{\sigma(k)})\beta(e_{\sigma(k+1)},\ldots,e_{\sigma(k+l)})$$

where Σ' denotes the sum over all *shuffles*; that is, permutations σ of $\{1,2,\ldots,k+l\}$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$. The basic properties of the operation $\wedge$ are given in the following.

2.3.5 Proposition. *For $\alpha \in T_k^0(E)$, $\beta \in T_l^0(E)$, and $\gamma \in T_m^0(E)$, we have*

(*i*) $\alpha\wedge\beta = A\alpha\wedge\beta = \alpha\wedge A\beta$;

(*ii*) $\wedge$ *is bilinear;*

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(*iii*) $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$;
(*iv*) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

Proof. For (i), first note that if $\sigma \in S_k$ and $\sigma t(e_1, \ldots, e_k) = t(e_{\sigma(1)}, \ldots, e_{\sigma(k)})$, then $A(\sigma t) = (\mathit{sign}\,\sigma) At$ for

$$A(\sigma t)(e_1, \ldots, e_k) = \frac{1}{k!} \sum_{\rho \in S_k} (\mathit{sign}\ \rho) t(e_{\rho\sigma(1)}, \ldots, e_{\rho\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\rho \in S_k} (\mathit{sign}\ \sigma)(\mathit{sign}\ \rho\sigma) t(e_{\rho\sigma(1)}, \ldots, e_{\rho\sigma(k)})$$

$$= (\mathit{sign}\ \sigma) At(e_1, \ldots, e_k)$$

since $\rho \mapsto \rho\sigma$ is a bijection. Then,

$$A(A\alpha \otimes \beta)(e_1, \ldots, e_k, \ldots, e_{k+l}) = A(A\alpha(e_1, \ldots, e_k)\beta(e_{k+1}, \ldots, e_{k+l}))$$

$$= A\left(\frac{1}{k!} \sum_{\tau \in S_k} \mathit{sign}\,\tau \alpha(e_{\tau(1)}, \ldots, e_{\tau(k)}) \beta(e_{k+1}, \ldots, e_{k+l}) \right)$$

$$= A \frac{1}{k!} \sum_{\tau \in S_k} (\mathit{sign}\,\tau)(\tau\alpha \otimes \beta)(e_1, \ldots, e_k, \ldots, e_{k+l})$$

$$= \frac{1}{k!} \sum_{\tau \in S_k} (\mathit{sign}\,\tau) A(\tau\alpha \otimes \beta)(e_1, \ldots, e_{k+l}) \quad \text{(linearity of } A\text{)}$$

$$= \frac{1}{k!} \sum_{\tau \in S_k} (\mathit{sign}\,\tau') A\tau'(\alpha \otimes \beta)(e_1, \ldots, e_{k+l})$$

where $\tau' \in S_{k+l}$,

$$\tau'(1, \ldots, k, \ldots, k+l) = (\tau(1), \ldots, \tau(k), k+1, \ldots, k+l)$$

so $\mathit{sign}\,\tau = \mathit{sign}\,\tau'$ and $\tau\alpha \otimes \beta = \tau'(\alpha \otimes \beta)$. Thus the above becomes

$$\frac{1}{k!} \sum_{\tau \in S_k} (\mathit{sign}\,\tau')(\mathit{sign}\,\tau') A(\alpha \otimes \beta)(e_1, \ldots, e_{k+l})$$

$$= A(\alpha \otimes \beta)(e_1, \ldots, e_{k+l}) \frac{1}{k!} \sum_{\tau \in S_k} 1$$

$$= A(\alpha \otimes \beta)(e_1, \ldots, e_{k+l})$$

Thus $A(A\alpha \otimes \beta) = A(\alpha \otimes \beta)$; that is, $(A\alpha) \wedge \beta = \alpha \wedge \beta$.

The other equality in (i) is similar.

Now (ii) is clear since $\otimes$ is bilinear and A is linear.

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For (iii), let $\sigma_0 \in S_{k+l}$ be given by $\sigma_0(1, \ldots, k+l) = (k+1, \ldots, k+l, 1, \ldots, k)$. Then $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}(e_1, \ldots, e_{k+l}) = \boldsymbol{\beta} \otimes \boldsymbol{\alpha}(e_{\sigma_0(1)}, \ldots, e_{\sigma_0(k+l)})$. Hence, by the proof of (i), $A(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) = (\mathit{sign}\ \sigma_0) A(\boldsymbol{\beta} \otimes \boldsymbol{\alpha})$. But $\mathit{sign}\ \sigma_0 = (-1)^{kl}$. Finally, (iv) follows from (i). ■

2.3.6 Definition. *The direct sum of the spaces $\Omega^k(E)$ $(k = 0, 1, 2 \ldots)$ together with its structure as a real vector space and multiplication induced by $\wedge$, is called the* ***exterior algebra*** *of E, or the* ***Grassmann algebra*** *of E.*

Using 2.3.5 and a simple induction argument, it follows that if $\boldsymbol{\alpha}_i$, $i = 1, \ldots, k$ are one-forms, then

$$(\boldsymbol{\alpha}_1 \wedge \cdots \wedge \boldsymbol{\alpha}_k)(e_1, \ldots, e_k) = \sum_{\sigma} (\mathit{sign}\, \sigma) \boldsymbol{\alpha}_1(e_{\sigma(1)}) \cdots \boldsymbol{\alpha}_k(e_{\sigma(k)})$$

We can now find a basis for $\Omega^k(E)$.

2.3.7 Proposition. *Let $n = \dim E$. Then for $k > n$, $\Omega^k(E) = \{0\}$, while for $0 < k \leqslant n$, $\Omega^k(E)$ has dimension $\binom{n}{k}$. The exterior algebra over E has dimension 2^n. Indeed, if $\hat{e} = (e_1, \ldots, e_n)$ is an ordered basis of E and $\hat{e}^* = (\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^n)$ its dual basis, a basis of $\Omega^k(E)$ is*

$$\left\{ \boldsymbol{\alpha}^{i_1} \wedge \cdots \wedge \boldsymbol{\alpha}^{i_k} \,\middle|\, 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n \right\}$$

Proof. First we show that the indicated wedge products span $\Omega^k(E)$. If $\boldsymbol{t} \in \Omega^k(E)$, then from 1.7.2 we know that

$$\boldsymbol{t} = \boldsymbol{t}(e_{i_1}, \ldots, e_{i_k}) \boldsymbol{\alpha}^{i_1} \otimes \cdots \otimes \boldsymbol{\alpha}^{i_k}$$

where the summation convention indicates that this should be summed over all choices of $i_1, \ldots, i_k$ between 1 and n, not just the ordered ones of the proposition. Now if the linear operator A is applied to this sum, we have, since $\boldsymbol{t} \in \Omega^k(E)$,

$$\boldsymbol{t} = A\boldsymbol{t} = \boldsymbol{t}(e_{i_1}, \ldots, e_{i_k}) A(\boldsymbol{\alpha}^{i_1} \otimes \cdots \otimes \boldsymbol{\alpha}^{i_k})$$

so that

$$\boldsymbol{t}(f_1, \ldots, f_k) = \boldsymbol{t}(e_{i_1}, \ldots, e_{i_k}) \frac{1}{k!} \sum_{\sigma \in S_k} (\mathit{sign}\, \sigma)(\boldsymbol{\alpha}^{i_1} \otimes \cdots \otimes \boldsymbol{\alpha}^{i_k})(f_{\sigma(1)}, \ldots, f_{\sigma(k)})$$

$$= \boldsymbol{t}(e_{i_1}, \ldots, e_{i_k}) \frac{1}{k!} (\boldsymbol{\alpha}^{i_1} \wedge \cdots \wedge \boldsymbol{\alpha}^{i_k})(f_1, \ldots, f_k)$$

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by the above remark. Therefore,

$$t = t(e_{i_1},\ldots,e_{i_k})\frac{1}{k!}\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}$$

The sum still runs over all choices of the $i_1,\ldots,i_k$ and we want only distinct, ordered ones. However, since t is skew symmetric, the coefficient $t(e_{i_1},\ldots,e_{i_k})$ is 0 if $i_1,\ldots,i_k$ are not distinct. If they are distinct and $\sigma \in S_k$, then

$$t(e_{i_1},\ldots,e_{i_k})\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k} = t(e_{\sigma(i_1)},\ldots,e_{\sigma(i_k)})\alpha^{\sigma(i_1)}\wedge\cdots\wedge\alpha^{\sigma(i_k)}$$

since both t and the wedge product change by a factor of *sign* σ. [Use 2.3.5(iii), where α and β are one-forms.] Since there are $k!$ of these rearrangements, we are left with

$$t = \sum_{i_1<\cdots<i_k} t(e_{i_1},\ldots,e_{i_k})\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}$$

Secondly, we show

$$\{\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k} \mid i_1<\cdots<i_k\}$$

are linearly independent. Suppose that

$$\sum_{i_1<\cdots<i_k} t_{i_1\cdots i_k}\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k} = 0$$

For fixed $i_1',\ldots,i_k'$, let $j_{k+1}',\ldots,j_n'$ denote the complementary set of indices, $j_{k+1}' < \cdots < j_n'$. Then

$$\sum_{i_1<\cdots<i_k} t_{i_1\cdots i_k}\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}\wedge\alpha^{j_{k+1}'}\wedge\cdots\wedge\alpha^{j_n'} = 0$$

However, this reduces to

$$t_{i_1'\cdots i_k'}\alpha^1\wedge\cdots\wedge\alpha^n = 0$$

But $\alpha^1\wedge\cdots\wedge\alpha^n \neq 0$, as $\alpha^1\wedge\cdots\wedge\alpha^n(e_1,\ldots,e_n) = 1$. Hence

$$t_{i_1'\cdots i_k'} = 0$$

The proposition now follows. ■

2.3.8 Definition. *The nonzero elements of the one-dimensional space $\Omega^n(E)$ are called* ***volume elements****. If ω_1 and ω_2 are volume elements, we say ω_1 and ω_2 are* ***equivalent*** *iff there is a $c>0$ such that $\omega_1 = c\omega_2$. An equivalence class of volume elements on E is called an* ***orientation*** *on E.*

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We shall see shortly the close relationship between volume elements and determinants.

2.3.9 Proposition. *Let $\alpha_1,\ldots,\alpha_k \in E^*$. Then $\alpha_1,\ldots,\alpha_k$ are linearly dependent iff $\alpha_1\wedge\cdots\wedge\alpha_k=0$.*

Proof. If $\alpha_1,\ldots,\alpha_k$ are linearly dependent, then

$$\alpha_i = \sum_{j\neq i} c_j\alpha_j$$

for some i. Then, since $\alpha\wedge\alpha=0$, we see $\alpha_1\wedge\cdots\wedge\alpha_k=0$. Conversely, if $\alpha_1,\ldots,\alpha_k$ are linearly independent, extend to a basis $\alpha_1,\ldots,\alpha_n$. Then $\alpha_1\wedge\cdots\wedge\alpha_n\neq 0$, by 2.3.7 and hence $\alpha_1\wedge\cdots\wedge\alpha_k\neq 0$. ■

2.3.10 Proposition. *Let $\dim(E)=n$ and $\varphi\in L(E,E)$. Then there is a unique constant $\det\varphi$, called the* ***determinant*** *of φ, such that $\varphi^*: \Omega^n(E)\to\Omega^n(E)$, defined by $\varphi^*\omega(e_1,\ldots,e_n)=\omega(\varphi(e_1),\ldots,\varphi(e_n))$ satisfies $\varphi^*\omega=(\det\varphi)\omega$ for all $\omega\in\Omega^n(E)$.*

Proof. Clearly $\varphi^*: \Omega^n(E)\to\Omega^n(E)$ is a linear mapping. But, from 2.3.7, $\Omega^n(E)$ is one-dimensional so that if ω_0 is a basis and $\omega=c\omega_0$, $\varphi^*\omega=c\varphi^*\omega_0=b\omega$ for some constant b, clearly unique. ■

It is easy to see that this definition of determinant is the usual one (Exercise 2.3B.) However, it has the advantage of suggesting the proper global definition (Sect. 2.5), as well as making its basic properties trivial, as follows.

2.3.11 Proposition. *Let $\varphi,\psi\in L(E,E)$. Then*

(*i*) *$\det(\varphi\circ\psi)=(\det\varphi)(\det\psi)$;*
(*ii*) *if φ is the identity, $\det\varphi=1$;*
(*iii*) *φ is an isomorphism iff $\det\varphi\neq 0$, and in this case $\det(\varphi^{-1})=(\det\varphi)^{-1}$.*

Proof. For (i), $(\varphi\circ\psi)^*\omega=\det(\varphi\circ\psi)\omega$, but $(\varphi\circ\psi)^*\omega=\psi^*\circ\varphi^*\omega$ as we see from the definitions as in 1.7.17. Hence, $(\varphi\circ\psi)^*\omega=\psi^*(\det\varphi)\omega=(\det\psi)(\det\varphi)\omega$ and (i) follows. (ii) follows at once from the definition. For (iii), suppose φ is an isomorphism with inverse φ^{-1}. Then, by (i) and (iii), $1=\det(\varphi\circ\varphi^{-1})=(\det\varphi)(\det\varphi^{-1})$, and, in particular, $\det\varphi\neq 0$. Conversely, if φ is not an isomorphism there is an $e_1\neq 0$ so $\varphi(e_1)=0$ (Exercise 1.2B). Extend to a basis $e_1,e_2,\ldots,e_n$. Then for all n-forms ω, $\varphi^*\omega(e_1,\ldots,e_n)=\omega(0,\varphi(e_2),\ldots,\varphi(e_n))=0$. Hence, $\det\varphi=0$. ■

Recall from Chapter 1 that there is a unique vector space topology on $L(E,E)$ since it is finite-dimensional. One convenient norm giving this

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topology, which was used earlier in 1.7.7, is the following operator norm:

$$\|\varphi\| = sup\{\|\varphi(e)\| \,|\, \|e\|=1\} = sup\left\{ \frac{\|\varphi(e)\|}{\|e\|} \,\middle|\, e \neq 0 \right\}$$

where $\|e\|$ is a norm on E. (See Exercise 1.2A). Hence, for any $e \in E$,

$$\|\varphi(e)\| \leqslant \|\varphi\| \, \|e\|$$

2.3.12 Proposition. *det: $L(E,E) \to R$ is continuous.*

Proof. Note that

$$\|\omega\| = sup\{|\omega(e_1,\ldots,e_n)| \,|\, \|e_1\| = \cdots = \|e_n\| = 1\}$$
$$= sup\{|\omega(e_1,\ldots,e_n)| / \|e_1\| \cdots \|e_n\| \,|\, e_1,\ldots,e_n \neq 0\}$$

is a norm on $\Omega^n(E)$ and $|\omega(e_1,\ldots,e_n)| \leqslant \|\omega\| \, \|e_1\| \cdots \|e_n\|$. Then, for $\varphi,\psi \in L(E,E)$,

$$|det\,\varphi - det\,\psi| \|\omega\| = \|\varphi^*\omega - \psi^*\omega\|$$
$$= sup\{|\omega(\varphi(e_1),\ldots,\varphi(e_n)) - \omega(\psi(e_1),\ldots,\psi(e_n))| \,|\, \|e_1\| = \cdots = \|e_n\| = 1\}$$
$$\leqslant sup\{|\omega(\varphi(e_1)-\psi(e_1),\varphi(e_2),\ldots,\varphi(e_n)| + \cdots$$
$$+ |\omega(\psi(e_1),\psi(e_2),\ldots,\varphi(e_n)-\psi(e_n))| \,|\, \|e_1\| = \cdots = \|e_n\| = 1\}$$
$$\leqslant \|\omega\| \|\varphi-\psi\| \{\|\varphi\|^{n-1} + \|\varphi\|^{n-2}\|\psi\| + \cdots + \|\psi\|^{n-1}\}$$
$$\leqslant \|\omega\| \|\varphi-\psi\| (\|\varphi\| + \|\psi\|)^{n-1}$$

Consequently, $|det\,\varphi - det\,\psi| \leqslant \|\varphi-\psi\|(\|\varphi\|+\|\psi\|)^{n-1}$ and the result follows. ∎

In 1.3.14 and 1.7.7 we saw that the isomorphisms are an open subset of $L(E,F)$. Using the determinant, we can give a simpler proof in the finite-dimensional case.

2.3.13 Proposition. *Suppose E and F are finite-dimensional and let $GL(E,F)$ denote those $\varphi \in L(E,F)$ that are isomorphisms. Then $GL(E,F)$ is an open subset of $L(E,F)$.*

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Proof. If $GL(\boldsymbol{E},\boldsymbol{F})=\varnothing$, the conclusion is true. If not, there is an isomorphism $\psi\in GL(\boldsymbol{E},\boldsymbol{F})$. A map φ in $L(\boldsymbol{E},\boldsymbol{F})$ is an isomorphism if and only if $\psi^{-1}\varphi$ is also. This happens precisely when $det(\psi^{-1}\varphi)\neq 0$. Therefore, $GL(\boldsymbol{E},\boldsymbol{F})$ is the inverse image of $\boldsymbol{R}\backslash\{0\}$ under the map taking φ to $det(\psi^{-1}\varphi)$. Since this is continuous and $\boldsymbol{R}\backslash\{0\}$ is open, $GL(\boldsymbol{E},\boldsymbol{F})$ is also open. ■

In order to define pull-back $\varphi^*\boldsymbol{t}$ or push-forward $\varphi_*\boldsymbol{t}$ of a general tensor $\boldsymbol{t}$ by a map φ, φ needs to be a diffeomorphism. For covariant tensors, however, pull-back makes sense if φ is merely a C^1 map. On the vector space level, this goes as follows.

2.3.14 Definition. *Let $\varphi\in L(\boldsymbol{E},\boldsymbol{F})$. For $\boldsymbol{\alpha}\in T^0_k(\boldsymbol{F})$ define the **pull-back** of $\boldsymbol{\alpha}$ by φ; $\varphi^*\alpha\in T^0_k(\boldsymbol{E})$ by $\varphi^*\boldsymbol{\alpha}(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k)=\boldsymbol{\alpha}(\varphi(\boldsymbol{e}_1),\ldots,\varphi(\boldsymbol{e}_k))$. If $\varphi\in GL(\boldsymbol{E},\boldsymbol{F})$, we denote by φ_* the **push-forward** map defined in* 1.7.3.

2.3.15 Proposition. *Let $\varphi\in L(\boldsymbol{E},\boldsymbol{F})$, $\psi\in L(\boldsymbol{F},\boldsymbol{G})$. Then*

(*i*) $\varphi^*: T^0_k(\boldsymbol{F})\to T^0_k(\boldsymbol{E})$ *is linear, and* $\varphi^*(\Omega^k(\boldsymbol{F}))\subset\Omega^k(\boldsymbol{E})$;
(*ii*) $(\psi\circ\varphi)^*=\varphi^*\circ\psi^*$;
(*iii*) *If φ is the identity, so is φ^*;*
(*iv*) *If $\varphi\in GL(\boldsymbol{E},\boldsymbol{F})$, then $\varphi^*\in GL(T^0_k(\boldsymbol{F}),T^0_k(\boldsymbol{E}))$, $(\varphi^*)^{-1}=(\varphi^{-1})^*$ and $\varphi^*\Omega^k(\boldsymbol{F})=\Omega^k(\boldsymbol{E})$;*
(*v*) *If $\varphi\in GL(\boldsymbol{E},\boldsymbol{F})$, then $\varphi_*\in GL(T^0_k(\boldsymbol{E}),T^0_k(\boldsymbol{F}))$, $(\varphi^{-1})^*=\varphi_*$, and $(\varphi_*)^{-1}=(\varphi^{-1})_*$; if $\psi\in GL(\boldsymbol{F},\boldsymbol{G})$, $(\psi\circ\varphi)_*=\psi_*\circ\varphi_*$;*
(*vi*) *If $\boldsymbol{\alpha}\in\Omega^k(\boldsymbol{F})$, $\boldsymbol{\beta}\in\Omega^l(\boldsymbol{F})$, then $\varphi^*(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})=\varphi^*\boldsymbol{\alpha}\wedge\varphi^*\boldsymbol{\beta}$.*

Proof. It is evident that (i) follows at once from the definition. For (ii),

$$\begin{aligned}(\psi\circ\varphi)^*\boldsymbol{\alpha}(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k)&=\boldsymbol{\alpha}(\psi\circ\varphi(\boldsymbol{e}_1),\ldots,\psi\circ\varphi(\boldsymbol{e}_k))\\&=\psi^*\boldsymbol{\alpha}(\varphi(\boldsymbol{e}_1),\ldots,\varphi(\boldsymbol{e}_k))\\&=\varphi^*\circ\psi^*\boldsymbol{\alpha}(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k)\end{aligned}$$

Then (iii) is clear and (iv) follows from (ii) and (iii). For (v), $\varphi_*\boldsymbol{\beta}(\boldsymbol{f}_1,\ldots,\boldsymbol{f}_k)=\boldsymbol{\beta}(\varphi^{-1}\boldsymbol{f}_1,\ldots,\varphi^{-1}\boldsymbol{f}_k)=(\varphi^{-1})^*\boldsymbol{\beta}(\boldsymbol{f}_1,\ldots,\boldsymbol{f}_k)$ and $(\varphi_*)^{-1}=(\varphi^{-1})^{*-1}=\varphi^*=(\varphi^{-1})_*$. Finally, $\varphi^*(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{k+l})=\boldsymbol{\alpha}\wedge\boldsymbol{\beta}(\varphi\boldsymbol{e}_1,\ldots,\varphi\boldsymbol{e}_{k+l})=\varphi^*\boldsymbol{\alpha}\wedge\varphi^*\boldsymbol{\beta}(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{k+l})$. ■

As in Sect. 1.7, we can consider the exterior algebra on the fibers of a vector bundle as follows.

2.3.16 Definition. *Let $\varphi: U\times\boldsymbol{F}\to U'\times\boldsymbol{F}'$ be a local vector bundle map that is an isomorphism on each fiber. Then define $\varphi_*: U\times\Omega^k(\boldsymbol{F})\to U'\times\Omega^k(\boldsymbol{F}')$ by $(u,\omega)\mapsto(\varphi(u),\varphi_{u*}\omega)$, where φ_u is the second factor of φ (an isomorphism for each u).*

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2.3.17 Proposition. *If $\varphi: U\times F\to U'\times F'$ is a local vector bundle map that is an isomorphism on each fiber, then so is φ_*. Moreover, if φ is a local vector bundle isomorphism, so is φ_*.*

Proof. This is a special case of 1.7.9. ■

2.3.18 Definition. *Suppose $\pi: E\to B$ is a vector bundle. Define*

$$\omega^k(E)|A=\bigcup_{b\in A}\Omega^k(E_b)$$

where A is a subset of B and $E_b=\pi^{-1}(b)$ is the fiber over $b\in B$. Let $\omega^k(E)|B=\omega^k(E)$ and define $\omega^k(\pi): \omega^k(E)\to B$ by $\omega^k(\pi)(t)=b$ if $t\in\Omega^k(E_b)$.

2.3.19 Theorem. *Suppose $\{E|U_i,\varphi_i\}$ is a vector bundle atlas of π, where φ_i: $E|U_i\to U_i'\times F_i'$. Then $\{\omega^k(E)|U_i,\varphi_{i_*}\}$ is a vector bundle atlas of $\omega^k(\pi)$: $\omega^k(E)\to B$, where $\varphi_{i_*}: \omega^k(E)|U_i\to U_i'\times\Omega^k(F_i')$ is defined by $\varphi_{i_*}|E_b=(\varphi_i|E_b)_*$ (as in 2.3.16).*

Proof. We must verify (VBA 1) and (VBA 2) of 1.5.2: (VBA 1) is clear; for (VBA 2) let φ_i,φ_j be two charts on π, so that $\varphi_i\circ\varphi_j^{-1}$ is a local vector bundle isomorphism. (We may assume $U_i=U_j$.) But then from 2.3.15, $\varphi_{i_*}\circ\varphi_{j_*}^{-1}=(\varphi_i\circ\varphi_j^{-1})_*$, which is a local vector bundle isomorphism by 2.3.17. ■

Because of this theorem, the vector bundle structure of $\pi: E\to B$ induces naturally a vector bundle structure on $\omega^k(\pi)$: $\omega^k(E)\to B$, which is also Hausdorff, second countable, and of constant dimension. Hereafter $\omega^k(\pi)$ will denote this vector bundle.

EXERCISES

2.3A. If $k!$ is omitted in the definition of A (2.3.2), show that $\wedge$ fails to be associative.

2.3B. Show that, in terms of components, our definition of the determinant is the usual one.

2.3C. If α is a two-form and β is a one-form, show that

$$(\alpha\wedge\beta)(e_1,e_2,e_3)=\alpha(e_1,e_2)\beta(e_3)-\alpha(e_1,e_3)\beta(e_2)+\alpha(e_2,e_3)\beta(e_1)$$

2.3D. Show that if $e_1,\ldots,e_n$ is a basis of E and $\alpha^1,\ldots,\alpha^n$ is the dual basis, then $(\alpha^1\wedge\cdots\wedge\alpha^n)(e_1,\ldots,e_n)=1$.

2.4 CARTAN'S CALCULUS OF DIFFERENTIAL FORMS

We now specialize the exterior algebra of the preceding section to tangent bundles and develop a differential calculus that is special to this case. This is basic to the dual integral calculus of Sect. 2.6 and to the Hamiltonian mechanics of Chapter 3.

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If $\tau_M: TM \to M$ is the tangent bundle of a manifold M, let $\omega^k(M) = \omega^k(TM)$, and $\omega_M^k = \omega^k(\tau_M)$, so $\omega_M^k: \omega^k(M) \to M$ is the vector bundle of exterior k forms on the tangent spaces of M. Also, let $\Omega^0(M) = \mathfrak{F}(M)$, $\Omega^1(M) = \mathfrak{T}_1^0(M)$, and $\Omega^k(M) = \Gamma^\infty(\omega_M^k), k = 2, 3, \ldots$.

2.4.1 Proposition. *Regarding $\mathfrak{T}_k^0(M)$ as an $\mathfrak{F}(M)$ module, $\Omega^k(M)$ is an $\mathfrak{F}(M)$ submodule.*

Proof. If $t_1, t_2 \in \Omega^k(M)$ and $f \in \mathfrak{F}(M)$, we must show $f \otimes t_1 + t_2 \in \Omega^k(M)$. From 1.7.19, we have $f \otimes t_1 + t_2 \in \mathfrak{T}_k^0(M)$. But, by 2.3.1, $f \otimes t_1(m) + t_2(m) \in \Omega^k(T_m M)$ and the result follows. ■

2.4.2 Proposition. *If $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M), k, l = 0, 1, \ldots, n$, define $\alpha \wedge \beta: M \to \omega^{k+l}(M)$ by $(\alpha \wedge \beta)(m) = \alpha(m) \wedge \beta(m)$. Then $\alpha \wedge \beta \in \Omega^{k+l}(M)$, and $\wedge$ is bilinear and associative.*

Proof. First, $\wedge$ is bilinear and associative by 2.3.5. To show $\alpha \wedge \beta$ is of class C^∞, consider the local representative of $\alpha \wedge \beta$ in natural charts. This is a map of the form $(\alpha \wedge \beta)_\varphi = \boldsymbol{B} \circ (\alpha_\varphi \times \beta_\varphi)$, with $\alpha_\varphi, \beta_\varphi, C^\infty$ and $\boldsymbol{B} = \wedge$, which is bilinear. Thus $(\alpha \wedge \beta)_\varphi$ is C^∞ by Leibniz' rule. ■

2.4.3 Definition. *Let $\Omega(M)$ denote the direct sum of $\Omega^k(M)$, $k = 0, 1, \ldots, n$, together with its structure as an (infinite-dimensional) real vector space and with the multiplication $\wedge$ extended componentwise to $\Omega(M)$. We call $\Omega(M)$ the* ***algebra of exterior differential forms*** *on M. Elements of $\Omega^k(M)$ are called* ***k-forms****. In particular, elements of $\mathfrak{X}^*(M)$ are called* ***one-forms****.*

Note that we generally regard $\Omega(M)$ as a real vector space rather than an $\mathfrak{F}(M)$ module [as with $\mathfrak{T}(M)$]. The reason is that $\mathfrak{F}(M) = \Omega^0(M)$ is included in the direct sum, and $f \wedge \alpha = f \otimes \alpha = f\alpha$.

2.4.4 Notation. *Let (U, φ) be a chart on a manifold M with $U' = \varphi(U) \subset \boldsymbol{R}^n$. Let $\boldsymbol{e}_i$ denote the standard basis of $\boldsymbol{R}^n$ and let $\underline{e}_i(u) = T_{\varphi(u)}\varphi^{-1}(\varphi(u), \boldsymbol{e}_i)$. Similarly let $\boldsymbol{\alpha}^i$ denote the dual basis of $\boldsymbol{e}_i$ and $\underline{\alpha}^i(u) = (T_u\varphi)^*(\varphi(u), \boldsymbol{\alpha}^i)$. [Thus, for each $u \in U, \underline{e}_i(u)$ and $\underline{\alpha}^i(u)$ are dual bases of the fiber $T_u M$.] Then if $\varphi(u) = (x^1(u), \ldots, x^n(u)) \in \boldsymbol{R}^n$, we define*

$$\frac{\partial f}{\partial x^i} = L_{\underline{e}_i} f = \frac{\partial f_\varphi}{\partial y^i} \circ \varphi$$

at points $u \in U$.

With these notations, we see $dx^i(u) = \underline{\alpha}^i(u)$, for

$$dx^i(u)(\underline{e}_j(u)) = P_2 T_u x^i \circ T_{\varphi(u)}\varphi^{-1}(\varphi(u), \boldsymbol{e}_j) = P_2 T_u (x^i \circ \varphi^{-1})(\varphi(u), \boldsymbol{e}_j)$$

$$= D(x^i \circ \varphi^{-1})(\varphi(u)) \cdot \boldsymbol{e}_j = \delta_j^i$$

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Hence,

$$df(u) = df(\underline{e}_i)\underline{\alpha}^i(u) = \frac{\partial f}{\partial x^i}(u)\, dx^i(u)$$

Thus the components of the differential df are the partial derivatives $\partial f/\partial x^i$.

Also, for each $t \in \mathfrak{T}^r_s(U)$ we have

$$t(u) = t^{i_1 \cdots i_r}_{j_1 \cdots j_s}(u)\underline{e}_{i_1} \otimes \cdots \otimes \underline{e}_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

and for each $\omega \in \Omega^k(U)$

$$\omega(u) = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k}(u)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}(u)$$

where

$$t^{i_1 \cdots i_r}_{j_1 \cdots j_s} = t\left(dx^{i_1}, \ldots, dx^{i_r}, \underline{e}_{j_1}, \ldots, \underline{e}_{j_s}\right)$$

and

$$\omega_{i_1 \cdots i_k} = \omega(\underline{e}_{i_1}, \ldots, \underline{e}_{i_k})$$

The extension of d to $\Omega^k(M)$ is given by the following.

2.4.5 Theorem. *Let M be a manifold. Then there is a unique family of mappings $\mathbf{d}^k(U)\colon \Omega^k(U) \to \Omega^{k+1}(U)$ ($k = 0, 1, 2, \ldots, n$, and U is open in M), which we merely denote by $\mathbf{d}$, called the **exterior derivative** on M, such that*

(*i*) *$\mathbf{d}$ is a $\wedge$ **antiderivation**. That is, $\mathbf{d}$ is $\mathbf{R}$ linear and for $\alpha \in \Omega^k(U)$, $\beta \in \Omega^l(U)$,*

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta$$

(*ii*) *If $f \in \mathfrak{F}(U)$, $\mathbf{d}f = df$ (as defined in 2.2.1);*

(*iii*) *$\mathbf{d} \circ \mathbf{d} = \mathbf{0}$ (that is, $\mathbf{d}^{k+1}(U) \circ \mathbf{d}^k(U) = \mathbf{0}$);*

(*iv*) *$\mathbf{d}$ is **natural with respect to restrictions**; that is, if $U \subset V \subset M$ are open and $\alpha \in \Omega^k(V)$, then $\mathbf{d}(\alpha | U) = (\mathbf{d}\alpha)| U$, or the following diagram commutes:*

$$\begin{array}{ccc} \Omega^k(V) & \xrightarrow{\;|U\;} & \Omega^k(U) \\ \mathbf{d}\downarrow & & \downarrow\mathbf{d} \\ \Omega^{k+1}(V) & \xrightarrow[\;|U\;]{} & \Omega^{k+1}(U) \end{array}$$

As in Sect. 2.2, condition (iv) means that $\mathbf{d}$ is a local operator.

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Proof. We first establish uniqueness. Using (iv) it is sufficient to consider the local case $\omega \in \Omega^k(U)$; $U \subset M$. By $\boldsymbol{R}$ linearity, it is sufficient to consider the case in which ω has the form $\omega = f_0\, df_1 \wedge \cdots \wedge df_k$, where $f_i \in \mathcal{F}(U)$. Hence, from (i), (ii), and (iii), $\boldsymbol{d}\omega = df_0 \wedge df_1 \wedge \cdots \wedge df_k$ and thus, $\boldsymbol{d}\omega$ is uniquely determined.

For existence we may again suppose $\omega = f_0\, df_1 \wedge \cdots \wedge df_k$ in some chart, and define $\boldsymbol{d}\omega = df_0 \wedge df_1 \wedge \cdots \wedge df_k$, which is independent of the chart (exercise). Then (ii) and (iv) are clear, as is $\boldsymbol{R}$ linearity. To prove (i), note that if $\rho = g_0\, dg_1 \wedge \cdots \wedge dg_l$, then

$$\begin{aligned}
\boldsymbol{d}(\omega \wedge \rho) &= \boldsymbol{d}(f_0 g_0) \wedge df_1 \wedge \cdots \wedge df_k \wedge dg_1 \wedge \cdots \wedge dg_l \\
&= g_0\, df_0 \wedge df_1 \wedge \cdots \wedge df_k \wedge dg_1 \wedge \cdots \wedge dg_l \\
&\quad + f_0\, dg_0 \wedge df_1 \wedge \cdots \wedge df_k \wedge dg_1 \wedge \cdots \wedge dg_l \\
&= \boldsymbol{d}\omega \wedge \rho + (-1)^k \omega \wedge \boldsymbol{d}\rho
\end{aligned}$$

Finally, for (iii), it is clearly sufficient to verify $\boldsymbol{d} \circ \boldsymbol{d}f = 0$ for functions. But in a local chart $df(u) = Df(u) \cdot e_i\, dx^i$ so that

$$\begin{aligned}
\boldsymbol{d} \circ df(u) &= DDf(u) \cdot (e_i, e_j)\, dx^j \wedge dx^i \\
&= \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i = 0
\end{aligned}$$

by symmetry of the mixed partial derivatives. ■

2.4.6 Corollary. *Let $\omega \in \Omega^k(U)$, where $U \subset \boldsymbol{E}$ (open). Then*

$$\boldsymbol{d}\omega(u)(e_0, \ldots, e_k) = \sum_{i=0}^{k} (-1)^i D\omega(u) \cdot e_i(e_0, \ldots, \hat{e}_i, \ldots, e_k)$$

where $\hat{e}_i$ denotes that e_i is deleted. Also, we denote elements (u, e) of TU merely by e, for brevity. [Note that $D\omega(u) \cdot e \in L^k(\boldsymbol{E}, \boldsymbol{R})$.]

Proof. First note that $\boldsymbol{d}$ defined this way is a map $\Omega^k(U) \to \Omega^{k+1}(U)$. Then it is sufficient to verify (i)–(iv) of 2.4.5. But $\boldsymbol{R}$ linearity, (ii), and (iv) are clear, and as $\wedge$ is bilinear, $D(\omega \wedge \rho) = \omega \wedge D\rho + D\omega \wedge \rho$, from which (i) readily follows. Finally, (iii) follows as in 2.4.5. ■

2.4.7 Definition. *Suppose $F: M \to N$ is a C^∞ mapping of manifolds. For $\omega \in \Omega^k(N)$, define $F^*\omega$: $M \to \omega^k(M)$ by $F^*\omega(m) = (T_m F)^* \circ \omega \circ F(m)$ (see* 2.3.14). *We say $F^*\omega$ is the* ***pull-back*** *of ω by F.*

Especially, note if $g \in \Omega^0(N)$, $F^*g = g \circ F$.

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2.4.8 Proposition. *Let $F: M \to N$ and $G: N \to W$ be C^∞ mappings of manifolds. Then*

(*i*) $F^*: \Omega^k(N) \to \Omega^k(M)$;
(*ii*) $(G \circ F)^* = F^* \circ G^*$;
(*iii*) *if $H: M \to M$ is the identity, then $H^*: \Omega^k(M) \to \Omega^k(M)$ is the identity;*
(*iv*) *if F is a diffeomorphism, then F^* is a vector bundle isomorphism and $(F^*)^{-1} = (F^{-1})^*$.*

Proof. Choose charts (U, φ), (V, ψ) of M and N so that $F(U) \subset V$, then $F_{\varphi\psi} = \psi \circ F \circ \varphi^{-1}$ is of class C^∞, as is $\omega_\psi = (T\psi)_* \circ \omega \circ \psi^{-1}$. Then

$$(T_{u'}T_{\varphi\psi})^* = (T_{u'}\varphi^{-1})^* \circ (T_u F)^* \circ (T_{F(u)}\psi)^* \qquad \text{(by 2.3.15)}$$

$$= (T_u\varphi)_* \circ (T_u F)^* \circ (T_{F(u)}\psi)^*$$

Hence the local representative of $F^*\omega$ is

$$(F^*\omega)_\varphi(u') = (T\varphi)_* \circ F^*\omega \circ \varphi^{-1}(u')$$

$$= (T_{u'}F_{\varphi\psi})^* \circ \omega_\psi \circ F_{\varphi\psi}(u')$$

which is of class C^∞ by the composite mapping theorem; $\boldsymbol{R}$ linearity is clear.

For (ii), we merely note that it holds for the local representatives by 2.3.15; (iii) follows at once from the definition; and (iv) follows in the usual way from (ii) and (iii) ■

As $F^*: \Omega^k(N) \to \Omega^k(M)$ is $\boldsymbol{R}$ linear, it induces a mapping on the direct sums, $F^*: \Omega(N) \to \Omega(M)$, which are differential algebras with $\wedge$ and $\boldsymbol{d}$.

2.4.9 Theorem. *Let $F: M \to N$ be of class C^∞. Then $F^*: \Omega(N) \to \Omega(M)$ is a homomorphism of differential algebras; that is,*

(*i*) $F^*(\psi \wedge \omega) = F^*\psi \wedge F^*\omega$, *and*
(*ii*) ***d* is natural with respect to mappings;** *that is, $F^*(\boldsymbol{d}\omega) = \boldsymbol{d}(F^*\omega)$, or the following diagram commutes:*

$$\begin{array}{ccc}
\Omega^k(N) & \xrightarrow{\;F^*\;} & \Omega^k(M) \\
{\scriptstyle \boldsymbol{d}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{d}} \\
\Omega^{k+1}(N) & \xrightarrow[\;F^*\;]{} & \Omega^{k+1}(M)
\end{array}$$

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Proof. We first consider $F^*(\psi\wedge\omega)$ when ψ is a function. Then

$$\begin{aligned} F^*(\psi\omega)(m) &= (T_mF)^*\circ\psi\omega\circ F(m)\\ &= (T_mF)^*\circ\left[(\psi\circ F)\cdot(\omega\circ F)\right](m)\\ &= \psi(F(m))F^*\omega(m) \end{aligned}$$

or $F^*(\psi\wedge\omega)=F^*\psi\wedge F^*\omega$, as $F^*\psi=\psi\circ F$ if $\psi\in\Omega^0(N)$. Then (i) follows immediately from 2.3.15(vi). For (ii) we shall show in fact that if $m\in M$, there is a neighborhood U of $m\in M$ such that $\mathbf{d}(F^*\omega|U)=(F^*\mathbf{d}\omega)|U$, which is sufficient, as F^* and $\mathbf{d}$ are both natural with respect to restriction. Let (V,φ) be a local chart at $F(m)$ and U a neighborhood of $m\in M$ with $F(U)\subset V$. Then for $\omega\in\Omega^k(V)$, we can write

$$\omega=\omega_{i_1\cdots i_k}\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$\mathbf{d}\omega=\partial_{i_0}\omega_{i_1\cdots i_k}\,dx^{i_0}\wedge\cdots\wedge dx^{i_k},\qquad \partial_{i_0}=\frac{\partial}{\partial x^{i_0}}$$

and by (i) above

$$F^*\omega|U=(F^*\omega_{i_1\cdots i_k})F^*dx^{i_1}\wedge\cdots\wedge F^*dx^{i_k}$$

But if $\psi\in\Omega^0(N)$, $\mathbf{d}(F^*\psi)=F^*\mathbf{d}\psi$ by the composite mapping theorem, so

$$\begin{aligned} \mathbf{d}(F^*\omega|U) &= F^*(\mathbf{d}\omega_{i_1\cdots i_k})\wedge F^*dx^{i_1}\wedge\cdots\wedge F^*dx^{i_k}\\ &= F^*(\mathbf{d}\omega)|U \end{aligned}$$

by (i) above. ■

2.4.10 Corollary. *The operator* $\mathbf{d}$ *is natural with respect to diffeomorphisms. That is, if* $F\colon M\to N$ *is a diffeomorphism, then* $F_*\mathbf{d}\omega=\mathbf{d}F_*\omega$, *or the following diagram commutes:*

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\;F_*\;} & \Omega^k(N)\\ \Big\downarrow \mathbf{d} & & \Big\downarrow \mathbf{d}\\ \Omega^{k+1}(M) & \xrightarrow[\;F_*\;]{} & \Omega^{k+1}(N) \end{array}$$

Proof. With F_* defined as $F_*=(F)^0_k$, we see that $F_*=(F^{-1})^*$. The result then follows from 2.4.9(ii). ■

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The next few propositions give some important relations between the Lie derivative and the exterior derivative.

2.4.11 Theorem. *Let $X \in \mathfrak{X}(M)$. Then $\boldsymbol{d}$ is natural with respect to $\boldsymbol{L}_X$. That is, for $\omega \in \Omega^k(M)$ we have $\boldsymbol{L}_X\omega \in \Omega^k(M)$ and $\boldsymbol{dL}_X\omega = \boldsymbol{L}_X\boldsymbol{d}\omega$, or the following diagram commutes:*

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\ \boldsymbol{L}_X\ } & \Omega^k(M) \\ \boldsymbol{d}\downarrow & & \downarrow\boldsymbol{d} \\ \Omega^{k+1}(M) & \xrightarrow[\ \boldsymbol{L}_X\ ]{} & \Omega^{k+1}(M) \end{array}$$

Proof. If $\alpha^1,\ldots,\alpha^k \in \Omega^1(M)$ we have

$$\boldsymbol{L}_X(\alpha^1\wedge\cdots\wedge\alpha^k) = \boldsymbol{L}_X\alpha^1\wedge\alpha^2\wedge\cdots\wedge\alpha^k + \cdots + \alpha^1\wedge\cdots\wedge\boldsymbol{L}_X\alpha^k$$

This follows from the fact that $\boldsymbol{L}_X$ is $\boldsymbol{R}$ linear and is a tensor derivation. Since locally $\omega \in \Omega^k(M)$ is a linear combination of such products, it readily follows that $\boldsymbol{L}_X\omega \in \Omega^k(M)$. For the second part, let (U,a,F) be a flow box at $m \in M$, so that from 2.2.20,

$$\boldsymbol{L}_X\omega(m) = \frac{d}{d\lambda}(F_\lambda^*\omega)(m)\bigg|_{\lambda=0}$$

But from 2.4.10 we have $F_\lambda^*\,\boldsymbol{d}\omega = \boldsymbol{d}(F_\lambda^*\omega)$. Then, since $\boldsymbol{d}$ is $\boldsymbol{R}$ linear, it commutes with $d/d\lambda$ and so $\boldsymbol{dL}_X\omega = \boldsymbol{L}_X\,\boldsymbol{d}\omega$. ■

The foregoing proof can also be carried out in terms of local representatives.

2.4.12 Definition. *Let M be a manifold, $X \in \mathfrak{X}(M)$, and $\omega \in \Omega^{k+1}(M)$. Then define $\boldsymbol{i}_X\omega \in \mathfrak{T}^0_k(M)$ by*

$$\boldsymbol{i}_X\omega(X_1,\ldots,X_k) = \omega(X, X_1,\ldots,X_k)$$

*If $\omega \in \Omega^0(M)$, we put $\boldsymbol{i}_X\omega = \boldsymbol{0}$. We call $\boldsymbol{i}_X\omega$ the **inner product** of X and ω.*

2.4.13 Theorem. *We have $\boldsymbol{i}_X\colon \Omega^k(M)\to\Omega^{k-1}(M)$, $k=1,\ldots,n$, and, for $\alpha\in\Omega^k(M)$, $\beta\in\Omega^l(M)$, $f\in\Omega^0(M)$,*

(*i*) *$\boldsymbol{i}_X$ is a $\wedge$ antiderivation. That is, $\boldsymbol{i}_X$ is $\boldsymbol{R}$ linear and $\boldsymbol{i}_X(\alpha\wedge\beta) = (\boldsymbol{i}_X\alpha)\wedge\beta + (-1)^k\alpha\wedge(\boldsymbol{i}_X\beta)$;*
(*ii*) *$\boldsymbol{i}_{fX}\alpha = f\boldsymbol{i}_X\alpha$;*
(*iii*) *$\boldsymbol{i}_X\boldsymbol{d}f = \boldsymbol{L}_Xf$;*
(*iv*) *$\boldsymbol{L}_X\alpha = \boldsymbol{i}_X\boldsymbol{d}\alpha + \boldsymbol{d}\boldsymbol{i}_X\alpha$;*
(*v*) *$\boldsymbol{L}_{fX}\alpha = f\boldsymbol{L}_X\alpha + \boldsymbol{d}f\wedge\boldsymbol{i}_X\alpha$.*

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Proof. That $i_X\alpha \in \Omega^{k-1}(M)$ follows at once from 2.2.8. For (i), $\boldsymbol{R}$ linearity is clear. For the second part of (i)

$$i_X(\alpha\wedge\beta)(X_2, X_3, \ldots, X_{k+l}) = (\alpha\wedge\beta)(X, X_2, \ldots, X_{k+l})$$

and

$$i_X\alpha\wedge\beta + (-1)^k\alpha\wedge i_X\beta = \frac{(k+l-1)!}{(k-1)!\,l!}A(i_X\alpha\otimes\beta)$$

$$+(-1)^k\frac{(k+l-1)!}{k!\,(l-1)!}A(\alpha\otimes i_X\beta)$$

But the sum over all permutations in the last term can be replaced by the sum over $\sigma\sigma_0$, where σ_0 is the permutation $(2, 3, \ldots, k+1, 1, k+2, \ldots, k+l)\mapsto(1, 2, 3, \ldots, k+l)$ whose sign is $(-1)^k$. Hence (i) follows. For (ii), we merely note α_X is linear, and (iii) is just the definition of L_Xf.

For (iv) we proceed by induction on k. First note that for $k=0$, (iv) reduces to (iii). Now assume that (iv) holds for k. Then a $k+1$ form may be written as $\Sigma df_i\wedge\omega_i$, where ω_i is a k form, in some neighborhood of $m\in M$. But $L_X(df\wedge\omega) = L_X\,df\wedge\omega + df\wedge L_X\omega$ and

$$\begin{aligned} i_X d(df\wedge\omega) + di_X(df\wedge\omega) &= -i_X(df\wedge d\omega) + d(i_Xdf\wedge\omega - df\wedge i_X\omega)\\ &= -i_X\,df\wedge d\omega + df\wedge i_X\,d\omega + di_X\,df\wedge\omega\\ &\quad + i_X\,df\wedge d\omega + df\wedge di_X\omega\\ &= df\wedge L_X\omega + dL_Xf\wedge\omega \end{aligned}$$

by our inductive assumption and (iii). Since $dL_Xf = L_X\,df$, the result follows.

Finally for (v) we have

$$\begin{aligned} L_{fX}\omega = i_{fX}\,d\omega + di_{fX}\omega &= f i_X\,d\omega + d(f i_X\omega)\\ &= f i_X\,d\omega + df\wedge i_X\omega + f\,di_X\omega\\ &= fL_X\omega + df\wedge i_X\omega \qquad \blacksquare \end{aligned}$$

The behavior of inner products under diffeomorphisms is given by the following.

2.4.14 Proposition. *Let M and N be manifolds and $f: M\to N$ a diffeomorphism. Then, if $\omega\in\Omega^k(N)$ and $X\in\mathfrak{X}(N)$, we have*

$$i_{f^*X}\,f^*\omega = f^*i_X\omega$$

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that is, inner products are natural with respect to diffeomorphisms; that is, the following diagram commutes:

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{f^*} & \Omega^k(M) \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_{f^*X}} \\ \Omega^{k-1}(N) & \xrightarrow[f^*]{} & \Omega^{k-1}(M) \end{array}$$

Similarly for $Y \in \mathfrak{X}(M)$ we have the following commutative diagram:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{f_*} & \Omega^k(N) \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle i_{f_*Y}} \\ \Omega^{k-1}(M) & \xrightarrow[f_*]{} & \Omega^{k-1}(N) \end{array}$$

Proof. Let $v_1,\ldots,v_{k-1} \in T_m(M)$ and $n = f(m)$. Then by 2.4.12 and 2.4.7

$$\begin{aligned} & i_{f^*X} f^*\omega(m)\cdot(v_1,\ldots,v_{k-1}) \\ &= f^*\omega(m)\cdot(f^*X(m), v_1,\ldots,v_{k-1}) \\ &= f^*\omega(m)\cdot(Tf^{-1}\circ X(n), v_1,\ldots,v_{k-1}) \\ &= \omega(n)\cdot(Tf\circ Tf^{-1}X(n), Tfv_1,\ldots,Tfv_{k-1}) \\ &= i_X\omega(n)\cdot(Tfv_1,\ldots,Tfv_{k-1}) \\ &= f^*i_X\omega(m)\cdot(v_1,\ldots,v_{k-1}) \quad \blacksquare \end{aligned}$$

The next proposition expresses d in terms of the Lie derivative (Palais [1963]).

2.4.15 Proposition. *Let $X_i \in \mathfrak{X}(M)$, $i = 0, \ldots, k$, and $\omega \in \Omega^k(M)$. Then we have*

(i) $$(L_{X_0}\omega)(X_1,\ldots,X_k) = L_{X_0}(\omega(X_1,\ldots,X_k)) - \sum_{i=1}^{k} \omega(X_1,\ldots,L_{X_0}X_i,\ldots,X_k)$$

(ii) $$d\omega(X_0, X_1,\ldots,X_k) = \sum_{i=0}^{k}(-1)^i L_{X_i}(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)) + \sum_{0\le i<j\le k}(-1)^{i+j}\omega(L_{X_i}(X_j), X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k)$$

where $\hat{X}_i$ denotes that X_i is deleted.

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Proof. Part (i) is exactly condition (DO 4) following 2.2.17. For (ii) we proceed by induction. For $k=0$, it is merely $d\omega(X_0)=L_{X_0}\omega$. Assume the formula for $k-1$. Then if $\omega \in \Omega^k(M)$, we have, by 2.4.13(iv),

$$\begin{aligned}
d\omega(X_0, X_1, \ldots, X_k) &= (i_{X_0}\, d\omega)(X_1, \ldots, X_k) \\
&= (L_{X_0}\omega)(X_1, \ldots, X_k) - \big(d(i_{X_0}\omega)\big)(X_1, \ldots, X_k) \\
&= L_{X_0}(\omega(X_1, \ldots, X_k)) \\
&\quad - \sum_1^k \omega(X_1, \ldots, L_{X_0}X_i, \ldots, X_k) \\
&\quad - (d i_{X_0}\omega)(X_1, \ldots, X_k) \qquad \text{(by (i))}
\end{aligned}$$

But $i_{X_0}\omega \in \Omega^{k-1}(M)$ and we may apply the induction assumption. This gives, after a simple permutation and 2.4.12,

$$\begin{aligned}
\big(d(i_{X_0}\omega)\big)(X_1, \ldots, X_k) &= \sum_{i=1}^{k} (-1)^{i-1} L_{X_i}(\omega(X_0, X_1, \ldots, \hat{X}_i, \ldots, X_k)) \\
&\quad - \sum_{1 \leqslant i<j \leqslant k} (-1)^{i+j} \omega\big(L_{X_i}X_j, X_0, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k\big)
\end{aligned}$$

Substituting this into the above easily yields the result. ■

2.4.16 Definition. *We call $\omega \in \Omega^k(M)$* ***closed*** *if $d\omega=0$, and* ***exact*** *if there is an $\alpha \in \Omega^{k-1}(M)$ such that $\omega = d\alpha$.*

2.4.17 Theorem. *(i) Every exact form is closed.*

(ii) ***(Poincaré lemma).*** *If ω is closed, then for each $m \in M$, there is a neighborhood U of m for which $\omega | U \in \Omega^k(U)$ is exact.*

Proof. Part (i) is clear since $d \circ d = 0$. Using a local chart and 2.4.9(ii) together with 2.4.5(iv), it is sufficient to consider the case $\omega \in \Omega^k(U)$, $U \subset E$ a disk about $0 \in E$, to prove (ii). On U we construct an R linear mapping $H: \Omega^k(U) \to \Omega^{k-1}(U)$ such that $d \circ H + H \circ d$ is the identity on $\Omega^k(U)$. This will give the result, for $d\omega = 0$ implies $d(H\omega) = \omega$.

For $e_1, \ldots, e_k \in E$ define

$$H\omega(u)(e_1, \ldots, e_{k-1}) = \int_0^1 t^{k-1} \omega(tu)(u, e_1, \ldots, e_{k-1})\, dt$$

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Then, by 2.4.6,

$$dH\omega(u)\cdot(e_1,\dots,e_k)=\sum_{i=1}^{k}(-1)^{i+1}DH\omega(u)\cdot e_i(e_1,\dots,\hat{e}_i,\dots,e_k)$$

$$=\sum_{i=1}^{k}(-1)^{i+1}\int_0^1 t^{k-1}\omega(tu)(e_i,e_1,\dots,\hat{e}_i,\dots,e_k)\,dt$$

$$+\sum_{i=1}^{k}(-1)^{i+1}\int_0^1 t^k D\omega(tu)\cdot e_i(u,e_1,\dots,\hat{e}_i,\dots,e_k)\,dt$$

(The interchange of D and $\int$ is permissible, as ω is smooth and bounded over $t\in[0,1]$.) However, we also have, by 2.4.6,

$$H\,d\omega(u)\cdot(e_1,\dots,e_k)=\int_0^1 t^k\,d\omega(tu)(u,e_1,\dots,e_k)\,dt$$

$$=\int_0^1 t^k D\omega(tu)\cdot u(e_1,\dots,e_k)\,dt$$

$$+\sum_{i=1}^{k}(-1)^{i}\int_0^1 t^k D\omega(tu)\cdot e_i(u,e_1,\dots,\hat{e}_i,\dots,e_k)\,dt$$

Hence

$$\left[dH\omega(u)+H\,d\omega(u)\right](e_1,\dots,e_k)=\int_0^1 kt^{k-1}\omega(tu)\cdot(e_1,\dots,e_k)\,dt$$

$$+\int_0^1 t^k D\omega(tu)\cdot u(e_1,\dots,e_k)\,dt$$

$$=\int_0^1 \frac{d}{dt}\left[t^k\omega(tu)\cdot(e_1,\dots,e_k)\right]dt$$

$$=\omega(u)\cdot(e_1,\dots,e_k)$$

which proves the assertion. ■

There is another proof of the Poincaré lemma that is useful to understand. This proof will help the reader master the proof of Darboux' theorem in Sect. 3.2, and is similar in spirit to the proof of Frobenius' theorem (2.2.26).

Alternative Proof of the Poincaré Lemma. We again let U be a ball about $\mathbf{0}$ in $\boldsymbol{E}$. Let, for $t>0$, $F_t(u)=tu$. Thus F_t is a diffeomorphism and, starting at

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$t=1$, is generated by the time-dependent vector field

$$X_t(u)=u/t$$

that is, $F_1(u)=u$ and $dF_t(u)/dt=X_t(F_t(u))$. Therefore, since ω is closed,

$$\begin{aligned}\frac{d}{dt}F_t^*\omega&=F_t^*\boldsymbol{L}_{X_t}\omega\\&=F_t^*(\boldsymbol{d}\boldsymbol{i}_{X_t}\omega)\\&=\boldsymbol{d}(F_t^*\boldsymbol{i}_{X_t}\omega)\end{aligned}$$

For $0<t_0\leqslant 1$, we get

$$\omega-F_{t_0}^*\omega=\boldsymbol{d}\int_{t_0}^1 F_t^*\boldsymbol{i}_{X_t}\omega\,dt$$

Letting $t_0\to 0$, we get $\omega=\boldsymbol{d}\beta$, where

$$\beta=\int_0^1 F_t^*\boldsymbol{i}_{X_t}\omega\,dt$$

Explicitly,

$$\beta_u(e_1,\ldots,e_{k-1})=\int_0^1 t^{k-1}\omega_{tu}(u,e_1,\ldots,e_{k-1})\,dt$$

(Note that this β agrees with that in the previous proof.) ■

See Exercise 2.4E for a relative Poincaré lemma.

It is not true that closed forms are always exact (for example, on a sphere). In fact, the quotient groups of closed forms by exact forms (called the de Rham cohomology groups of M) shed light on the manifold topology. A discussion may be found in Flanders [1963], Singer and Thorpe [1967], and in de Rham [1955].

In differential geometry the use of vector valued forms is important; that is, one replaces multilinear maps into $\boldsymbol{R}$ by multilinear maps into a vector space V. One can utilize the exterior calculus by taking the components of the form. For applications to geometry, see Kobayashi–Nomizu [1963], Chern [1972], or Spivak [1974].

The following table summarizes some of the important algebraic identities involving differential forms that have been obtained.

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Table 2.4-1

1. Vector fields on M with the bracket $[X,Y]$ form a Lie algebra; that is, $[X,Y]$ is real bilinear, skew symmetric, and Jacobi's identity holds:

$$[[X,Y],Z]+[[Z,X],Y]+[[Y,Z],X]=0.$$

2. For a diffeomorphism f, $f_*[X,Y]=[f_*X,f_*Y]$ and $(f\circ g)_*X=f_*g_*X$.
3. The forms on a manifold are a real associative algebra with $\wedge$ as multiplication. Furthermore, $\alpha\wedge\beta=(-1)^{kl}\beta\wedge\alpha$ for k and l forms α and β, respectively.
4. If f is a map, $f^*(\alpha\wedge\beta)=f^*\alpha\wedge f^*\beta$, $(f\circ g)^*\alpha=g^*f^*\alpha$.
5. $\mathbf{d}$ is a real linear map on forms and:

$$\mathbf{dd}\alpha=0,\ \mathbf{d}(\alpha\wedge\beta)=\mathbf{d}\alpha\wedge\beta+(-1)^k\alpha\wedge\mathbf{d}\beta \text{ for } \alpha \text{ a } k\text{-form.}$$

6. For α a k-form and $X_0,\ldots,X_k$ vector fields:

$$\mathbf{d}\alpha(X_0,\ldots,X_k)=\sum_{i=0}^{k}(-1)^iX_i\big(\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k)\big)$$

$$+\sum_{i<j}(-1)^{i+j}\alpha\big([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k\big)$$

7. For a map f, $f^*\mathbf{d}\alpha=\mathbf{d}f^*\alpha$.
8. (Poincaré lemma) If $\mathbf{d}\alpha=0$, then α is locally exact; that is, there is a neighborhood U about each point on which $\alpha=\mathbf{d}\beta$.
9. $\mathbf{i}_X\alpha$ is a real bilinear in X,α and for $h: M\to\mathbf{R}$, $\mathbf{i}_{hX}\alpha=h\mathbf{i}_X\alpha=\mathbf{i}_Xh\alpha$. Also $\mathbf{i}_X\mathbf{i}_X\alpha=0$, and

$$\mathbf{i}_X(\alpha\wedge\beta)=\mathbf{i}_X\alpha\wedge\beta+(-1)^k\alpha\wedge\mathbf{i}_X\beta.$$

10. For a diffeomorphism f, $f^*\mathbf{i}_X\alpha=\mathbf{i}_{f^*X}f^*\alpha$.
11. $\mathbf{L}_X\alpha=\mathbf{d}\mathbf{i}_X\alpha+\mathbf{i}_X\mathbf{d}\alpha$.
12. $\mathbf{L}_X\alpha$ is real bilinear in X,α and $\mathbf{L}_X(\alpha\wedge\beta)=\mathbf{L}_X\alpha\wedge\beta+\alpha\wedge\mathbf{L}_X\beta$.
13. For a diffeomorphism f, $f^*\mathbf{L}_X\alpha=\mathbf{L}_{f^*X}f^*\alpha$.
14. $(\mathbf{L}_X\alpha)(X_1,\ldots,X_k)=X(\alpha(X_1,\ldots,X_k))-\sum_{i=1}^k\alpha(X_1,\ldots,[X,X_i],\ldots,X_k)$.
15. Locally,

$$(\mathbf{L}_X\alpha)_x\cdot(v_1,\ldots,v_k)=D\alpha_x\cdot X(x)\cdot(v_1,\ldots,v_k)+\sum_{i=1}^k\alpha_x\cdot(v_1,\ldots,DX_x\cdot v_i,\ldots,v_k).$$

16. The following identities hold:

$$\mathbf{L}_{fX}\alpha=f\mathbf{L}_X\alpha+\mathbf{d}f\wedge\mathbf{i}_X\alpha$$

$$\mathbf{L}_{[X,Y]}\alpha=\mathbf{L}_X\mathbf{L}_Y\alpha-\mathbf{L}_Y\mathbf{L}_X\alpha$$

$$\mathbf{i}_{[X,Y]}\alpha=\mathbf{L}_X\mathbf{i}_Y\alpha-\mathbf{i}_Y\mathbf{L}_X\alpha$$

$$\mathbf{L}_X\mathbf{d}\alpha=\mathbf{d}\mathbf{L}_X\alpha$$

$$\mathbf{L}_X\mathbf{i}_X\alpha=\mathbf{i}_X\mathbf{L}_X\alpha$$

EXERCISES

2.4A. On S^1 find a closed one-form α that is not exact. What are the cohomology groups of S^1?

2.4B. Show that the following properties uniquely characterize $\mathbf{i}_X$:
(i) $\mathbf{i}_X: \Omega^k(M)\to\Omega^{k-1}(M)$ is a $\wedge$ antiderivation;
(ii) $\mathbf{i}_Xf=0$; $f\in\mathfrak{F}(M)$;
(iii) $\mathbf{i}_X\omega=\omega(X)$ for $\omega\in\Omega^1(M)$;
(iv) $\mathbf{i}_X$ is natural with respect to restrictions.
Hence show $\mathbf{i}_{[X,Y]}=\mathbf{L}_X\mathbf{i}_Y-\mathbf{i}_Y\mathbf{L}_X$. Finally, show $\mathbf{i}_X\circ\mathbf{i}_X=0$.

2.4C. If $\omega\in\Omega^k(M)$, and if, for some $f\in\mathfrak{F}(M)$, $f(m)\neq0$ for all $m\in M$ and $f\omega$ is

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exact, there is a $\theta \in \Omega^1(U)$ with $d\omega = \theta \wedge \omega$ and $d\omega \wedge \omega = 0$. Interpret as a necessary condition for integrability of a total differential equation. Such a function f is an *integrating factor* of ω. For a partial converse, see Flanders [1963, p. 94].

2.4D. Let $s: T^2M \to T^2M$ be the canonical involution of the second tangent bundle (see Exercise 1.6D).
(i) If X is a vector field on M, show that $s \circ TX$ is a vector field on TM.
(ii) If F_t is the flow of X, TF_t is a flow on TM generated by $s \circ TX$.
(iii) If μ is a one form on M, $\hat{\mu}: TM \to \mathbf{R}$ the corresponding function, and $w \in T^2M$, then show that

$$d\hat{\mu}(sw) = d\mu(\tau_{TM}(w), T\tau_M(w)) + d\hat{\mu}(w)$$

2.4E. Prove the following *relative Poincaré lemma*: Let ω be a closed k-form on a manifold M and let $N \subset M$ be a closed submanifold. Assume that the pull-back of ω to N is zero. Then there is a $(k-1)$-form α on a neighborhood of N such that $d\alpha = \omega$ and α vanishes on N. If ω vanishes on N, then α can be chosen so that all its first partial derivatives vanish on N. (*Hint:* Let φ_t be a homotopy of a neighborhood of N to N and construct an $\boldsymbol{H}$ operator as in the Poincaré lemma using φ_t.)

2.4F. (Angular Variables). Let S^1 denote the circle, $S^1 \approx \mathbf{R}/(2\pi) \approx \{z \in \mathbf{C} \mid |z| = 1\}$. Let $\gamma: \mathbf{R} \to S^1: x \mapsto e^{ix}$, be the exponential map. Show that γ induces an isomorphism $TS^1 \approx S^1 \times \mathbf{R}$. Let M be a manifold and let ω be an "angular variable," that is, a smooth map $\omega: M \to S^1$. Define $d\omega$, a one form on M by taking the $\mathbf{R}$-projection of $T\omega$. Show that (i) if $\omega: M \to S^1$, then $d^2\omega = 0$; and (ii) if $f: M \to N$ is smooth, then $f^*(d\omega) = d(f^*\omega)$, where $f^*\omega = \omega \circ f$.

2.5 ORIENTABLE MANIFOLDS

The purpose of this section is to globalize the definitions of orientation and determinant discussed in Sect. 2.3. This leads naturally to the definition of the divergence of a vector field. First, we discuss partitions of unity, which are used in some proofs of this section, and which are essential for the definition of the integral (Sect. 2.6).

2.5.1 Definitions. *If t is a tensor field on a manifold M, the* ***support*** *of t is the closure of the set of $m \in M$ for which $t(m) \neq 0$, and is denoted* supp t. *Also, we say t has* ***compact support*** *if* supp t *is compact in M.*

A collection of subsets $\{C_\alpha\}$ of a manifold M (or, more generally, a topological space) is called ***locally finite*** *if for each $m \in M$ there is a neighborhood U of m such that $U \cap C_\alpha = \varnothing$ except for finitely many indices α.*

2.5.2 Definition. *A* ***partition of unity*** *on a manifold M is a collection $\{(U_i, g_i)\}$, where*

(*i*) *$\{(U_i)\}$ is a locally finite open covering of M;*
(*ii*) *$g_i \in \mathcal{F}(M)$, $g_i(m) \geq 0$ for all $m \in M$, g_i has compact support, and* supp $g_i \subset U_i$ *for all i;*
(*iii*) *For each $m \in M$, $\Sigma_i g_i(m) = 1$.*

[*By* (*i*), *this is a finite sum.*]

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If $\mathcal{A}$ is an atlas on M, a ***partition of unity subordinate to*** *$\mathcal{A}$ is partition of unity $\{(U_i, g_i)\}$ such that each open set U_i is a restriction of a chart of $\mathcal{A}$ to an open subset of its domain.*

2.5.3 Theorem. *If $\mathcal{A}$ is an atlas of M, there is a partition of unity subordinate to $\mathcal{A}$.*

Proof. The proof of 1.1.21 shows the following. Let M be an n manifold and $\{W_\alpha\}$ be an open covering. Then there is a locally finite refinement consisting of charts $\{V_i, \phi_i\}$ such that $\phi_i(V_i)$ is the disk of radius 3, and such that $\phi_i^{-1}(D_1(0))$ cover M, where $D_1(0)$ is the unit disk, centered at the origin in the model space. Now let $\mathcal{A}$ be an atlas on M and let $\{V_i, \phi_i\}$ be a locally finite refinement with these properties. From 2.2.7 there is a nonzero function $h_i \in \mathcal{F}(M)$ whose support lies in V_i and $h_j \geqslant 0$. Let

$$g_i(u) = \frac{h_i(u)}{\Sigma_i h_i(u)}$$

(the sum is finite). These are the required functions. ■

Proof of the parenthetical statement in 2.2.7. More generally, we prove a smooth version of Urysohn's lemma (1.1.23). Let A and B be two closed sets. Since manifolds are normal (see 1.1.21, and 1.1.22), there is an atlas $\{U_\alpha, \phi_\alpha\}$ such that $U_\alpha \cap A \neq \varnothing$ implies $U_\alpha \cap B = \varnothing$. Let $\{V_i, g_i\}$ be a subordinate partition of unity and $h = \Sigma g_i$, where the sum is over those i for which $V_i \cap A \neq \varnothing$. Then h is one on A and zero on B. ■

2.5.4 Definition. *A* ***volume*** *on an n-manifold M is an n-form $\Omega \in \Omega^n(M)$ such that $\Omega(m) \neq 0$ for all $m \in M$; M is called* ***orientable*** *if there is a volume on M.*

Thus, Ω assigns an orientation, as defined in 2.3.8, to each fiber of TM.

2.5.5 Theorem. *Let M be a connected n-manifold. Then (i) M is orientable iff $\Omega^n(M)$, regarded as an $\mathcal{F}(M)$ module, is one-dimensional (has one generator);*

(ii) M is orientable iff M has an atlas $\{(U_i, \varphi_i)\}$, where $\varphi_i\colon U_i \to U_i' \subset \mathbf{R}^n$, such that the Jacobian determinant of the overlap maps is positive (the Jacobian determinant being the determinant of the derivative, a linear map from $\mathbf{R}^n$ into $\mathbf{R}^n$).

Proof. For (i) assume first that M is orientable, with a volume Ω. Let Ω' be any other element of $\Omega^n(M)$. Now each fiber of $\Omega^n(M)$ is one-dimensional, so we may define a map $f\colon M \to \mathbf{R}$ by

$$\Omega'(m) = f(m)\Omega(m)$$

We must show that $f \in \mathcal{F}(M)$. In local representation,

$$\Omega'(m) = \omega'(m)\, dx^{i_1} \wedge \cdots \wedge dx^{i_n}(m)$$

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and $\Omega(m)=\omega(m)\,dx^{i_1}\wedge\cdots\wedge dx^{i_n}(m)$. But $\omega(m)\neq 0$ for all $m\in M$. Hence $f(m)=\omega'(m)/\omega(m)$ is of class C^∞. Conversely, if $\Omega^n(M)$ is generated by Ω, then $\Omega(m)\neq 0$ for all $m\in M$ since each fiber is one-dimensional.

To prove (ii), let $\{(U_i,\varphi_i)\}$ be an atlas with $U_i'\cap \boldsymbol{R}^n$. Also, we may assume that all U_i' are connected by taking restrictions if necessary. Now $\varphi_{i*}\Omega=f_i dx^1\wedge\cdots\wedge dx^n=f_i\Omega_0$, where Ω_0 is the standard volume element on $\boldsymbol{R}^n$. By means of a reflection if necessary, we may assume that $f_i(u')>0$ ($f_i\neq 0$ since Ω is a volume). However, a continuous real valued function on a connected space which is not zero is always >0 or always <0. Hence, for overlap maps we have

$$\begin{aligned}(\varphi_i\circ\varphi_j^{-1})_* dx^1\wedge\cdots\wedge dx^n &= \varphi_{i_*}\circ\varphi_{j_*}^{-1} dx^1\wedge\cdots\wedge dx^n\\ &= \frac{f_i}{f_j\circ\varphi_j\circ\varphi_i^{-1}}\,dx^1\wedge\cdots\wedge dx^n\end{aligned}$$

But,

$$\psi^*(u)(\alpha^1\wedge\cdots\wedge\alpha^n)=D\psi(u)^*\cdot\alpha^1\wedge D\psi(u)^*\cdot\alpha^2\wedge\cdots\wedge D\psi(u)^*\cdot\alpha^n$$

where $D\psi(u)^*\cdot\alpha^1(e)=\alpha^1(D\psi(u)\cdot e)$. Hence, by definition of determinant we have

$$det\left(D\left(\varphi_j\circ\varphi_i^{-1}\right)(u)\right)=\frac{f_i(u)}{f_j\left[\varphi_j\circ\varphi_i^{-1}(u)\right]}>0$$

We leave as an exercise for the reader that the canonical isomorphism $L(\boldsymbol{E};\,\boldsymbol{E})\approx L(\boldsymbol{E}^*;\,\boldsymbol{E}^*)$, used above, does not affect determinants.

For the converse of (ii), let $\{(V_\alpha,\psi_\alpha)\}$ be an atlas with the given property, and $\{(U_i,\varphi_i,g_i)\}$ a subordinate partition of unity. Let

$$\Omega_i=\varphi_i^*(dx^1\wedge\cdots\wedge dx^n)\in\Omega^n(U_i)$$

and let

$$\tilde{\Omega}_i(m)=\begin{cases} g_i(m)\Omega_i(m) & \text{if } m\in U_i\\ 0 & \text{if } m\notin U_i\end{cases}$$

Since $supp\, g_i\subset U_i, \tilde{\Omega}_i\in\Omega^n(M)$. Then let

$$\Omega=\sum_i\tilde{\Omega}_i$$

Since this sum is finite in some neighborhood of each point, it is clear from local representatives that $\Omega\in\Omega^n(M)$. Finally, as the overlap maps have

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positive Jacobian determinant, then on $U_i \cap U_j$, $\Omega_i \neq 0$ and

$$\Omega_j = \varphi_j^*(dx^1_\wedge \cdots {}_\wedge dx^n) = \varphi_i^*(\varphi_j \circ \varphi_i^{-1})^*(dx^1 \wedge \cdots \wedge dx^n)$$

$$= \left[det\, D(\varphi_j \circ \varphi_i^{-1}) \circ \varphi_i \right] \varphi_i^*(dx^1 \wedge \cdots \wedge dx^n)$$

Since $\Sigma_j g_j = 1$, it is clear then that $\Omega(m) \neq 0$ for each $m \in M$. ■

Thus, if M is an orientable manifold, with volume Ω, 2.5.5(i) defines a map from $\Omega^n(M)$ into $\mathfrak{F}(M)$; namely, for each $\Omega' \in \Omega^n(M)$, there is a unique $f \in \mathfrak{F}(M)$ such that $\Omega' = f\Omega$.

2.5.6 Definition. *Let M be an orientable manifold. Two volumes Ω_1 and Ω_2 on M are called* ***equivalent*** *iff there is an $f \in \mathfrak{F}(M)$ with $f(m) > 0$ for all $m \in M$ such that $\Omega_1 = f\Omega_2$. (This is clearly an equivalence relation.) An* ***orientation*** *of M is an equivalence class $[\Omega]$ of volumes on M. An* ***oriented manifold,*** *$(M, [\Omega])$, is an orientable manifold M together with an orientation $[\Omega]$ on M.*

If $[\Omega]$ is an orientation of M, then $[-\Omega]$ (which is clearly another orientation) is called the ***reverse orientation.***

The next proposition tells us when $[\Omega]$ and $[-\Omega]$ are the only two orientations.

2.5.7 Proposition. *Let M be an orientable manifold. Then M is connected iff M has exactly two orientations.*

Proof. Suppose M is connected, and Ω, Ω' are two volumes with $\Omega' = f\Omega$. Since M is connected, and $f(m) \neq 0$ for all $m \in M$, $f(m) > 0$ for all m or else $f(m) < 0$ for all m. Thus Ω' is equivalent to Ω or $-\Omega$. Conversely, if M is not connected, let $U \neq \varnothing$ or M be a subset that is both open and closed. If Ω is a volume on M, define Ω' by

$$\Omega'(m) = \begin{cases} \Omega(m) & m \in U \\ -\Omega(m) & m \notin U \end{cases}$$

Obviously Ω' is a volume on M, and $\Omega' \notin [\Omega] \cup [-\Omega]$. ■

A simple example of a nonorientable manifold is the Möbius band (see Fig. 2.5-1

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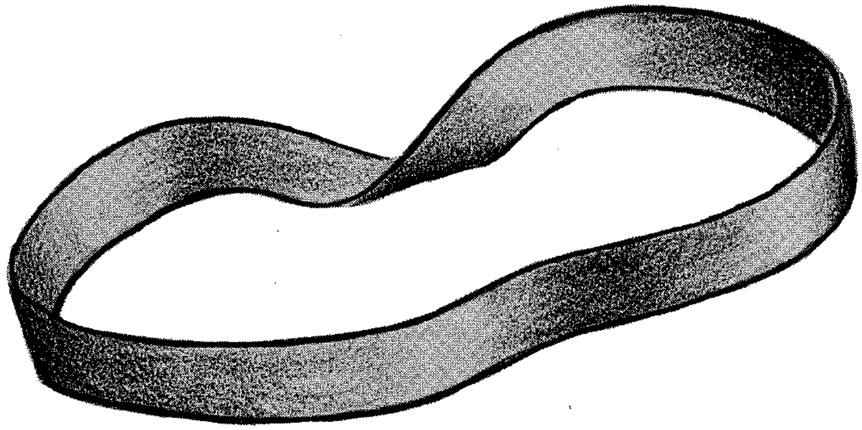

Figure 2.5-1

2.5.8 Proposition. *The equivalence relation in 2.5.6 is natural with respect to mappings and diffeomorphisms. That is, if $f: M \to N$ is of class C^∞, Ω_N and Ω'_N are equivalent volumes on N, and $f^*(\Omega_N)$ is a volume on M, then $f^*(\Omega'_N)$ is an equivalent volume. If f is a diffeomorphism and Ω_M and Ω'_M are equivalent volumes on M, then $f_*(\Omega_M)$ and $f_*(\Omega'_M)$ are equivalent volumes on N.*

Proof. This follows easily from the fact that

$$f^*(g\omega) = (g \circ f) f^*\omega$$

which implies

$$f_*(g\omega) = (g \circ f^{-1}) f_*\omega$$

when f is a diffeomorphism. ■

2.5.9 Definition. *Let M be an orientable manifold with orientation $[\Omega]$. A chart (U, φ) with $\varphi(U) = U' \subset \mathbf{R}^n$ is called **positively oriented** iff $\varphi_*(\Omega | U)$ is equivalent to the standard volume*

$$dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(U')$$

From 2.5.8 we see that the above definition does not depend on the choice of the representative from $[\Omega]$.

If M is orientable, we can find an atlas in which every chart has positive orientation by choosing an atlas of connected charts and, if a chart has negative orientation, by composing it with a reflection. Thus, in 2.5.5(ii), the atlas consists of positively oriented charts.

2.5.10 Definition. *Let V be a submanifold of an n-manifold M. We say V has **codimension** k iff V, considered as a manifold, has dimension $n-k$.*

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Now since a curve in V is also a curve in M, we can say $T_vV \subset T_vM$, and it is clear from Sect. 1.6 that the submanifold V has codimension k iff T_vV has dimension $n-k$ for each $v \in V$ iff for each $v \in V$ there is a vector space W_v of dimension k so that $T_vM = T_vV \oplus W_v$ (direct sum).

2.5.11 Proposition. *Suppose M is an orientable n-manifold and V is a submanifold of codimension k with trivial normal bundle. That is, there are C^∞ maps $N_i: V \to TM$, $i=1,\ldots,k$ such that $N_i(v) \in T_v(M)$, and $N_i(v)$ span a subspace W_v such that $T_vM = T_vV \oplus W_v$ for all $v \in V$. Then V is orientable.*

Proof. Let Ω be a volume on M. Form $\Omega | V: V \to \Omega^n(M)$. Let us first note that $\Omega | V$ is a smooth mapping of manifolds. (This was obvious earlier when we considered open submanifolds.) This follows at once by using charts with the submanifold property, where the local representative is a restriction to a subspace. Now define $\Omega_0: V \to \Omega^{n-k}(V)$ as follows: for

$$X_1, \ldots, X_{n-k} \in \mathfrak{X}(V)$$

put

$$\Omega_0(v)(X_1(v), \ldots, X_{n-k}(v)) = \Omega(v)(N_1(v), \ldots, N_k(v), X_1(v), \ldots, X_{n-k}(v))$$

(analogous to an inner product; however N_i are not vector fields on M). It is clear that $\Omega_0(v) \neq 0$ for all v. It remains only to show that Ω_0 is smooth, but this follows from the fact that $\Omega | V$ is smooth. ■

For some of the following proofs it will be convenient to use a Riemannian metric.

2.5.12 Definition. *A **Riemannian metric** on a manifold M is a tensor $g \in \mathfrak{T}^0_2(M)$ such that for all $m \in M$, $g(m)$ is symmetric and positive-definite.*

2.5.13 Proposition. *On any manifold there exists a Riemannian metric.*

Proof. Let $\{(U_i, \varphi_i, h_i)\}$ be a partition of unity on M, with $U_i' = \varphi_i(U_i) \subset \mathbf{R}^n$. If H_i is the standard Riemannian metric on U_i',

$$H_i(u)(v,w) = \sum v^i w^i$$

let $g_i \in \mathfrak{T}^0_2(m)$ be defined by

$$g_i(m) = \begin{cases} h_i(m)(\varphi_i^{-1})_* H_i(m) & \text{if } m \in U_i \\ 0 & \text{if } m \notin U_i \end{cases}$$

Then $g = \Sigma_i g_i$ is a Riemannian metric on M. ■

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Recall that we include second countable in our definition of a manifold. It is interesting that a manifold which admits a Riemannian metric (or a connection) must be second countable (see Abraham [1963]).

Note that if $g \in \mathfrak{T}^0_2(M)$, we may identify g with an $\mathfrak{F}$ linear mapping $g^\flat \in L(\mathfrak{X}, \mathfrak{X}^*)$ and if g is a Riemannian metric, obviously $g^\flat$ is an isomorphism. In this case we write $g^\sharp = (g^\flat)^{-1}$, and the maps $g^\sharp$ and $g^\flat$ are called raising and lowering indices, respectively.

2.5.14 Definition. *Let M be a manifold with a Riemannian metric g. For $f \in \mathfrak{F}(M)$, $grad f = g^\sharp(df)$ is called the **gradient** of f. Thus, $grad f \in \mathfrak{X}(M)$. In local coordinates, if $g_{ij} = g(e_i, e_j)$ and g^{ij} is the inverse matrix, then one checks that*

$$(grad\, f)^i = g^{ij} \frac{\partial f}{\partial x^j}$$

The above machinery allows us to obtain the following consequence of 2.5.11.

2.5.15 Theorem. *Suppose M is an orientable manifold, $H \in \mathfrak{F}(M)$ and $c \in \mathbf{R}$ is a regular value of H. Then $V = H^{-1}(c)$ is an orientable submanifold of M of codimension one, if it is nonempty.*

Proof. Suppose c is regular value of H and $H^{-1}(c) = V \neq \phi$. Then V is a submanifold of codimension one. Let g be a Riemannian metric on M and $N = grad(H)|V$. Then $N(v) \notin T_vV$ for $v \in V$, because T_vV is the kernel of $dH(v)$, and $dH(v)[N(v)] = g(N, N)(v) > 0$ as $dH(v) \neq 0$ by hypothesis. Then 2.5.11 applies, and so V is orientable. ■

Thus if we interpret V as the "energy surface," we see that it is an oriented submanifold for "almost all" energy values (Sard's theorem).

Let us now examine the effect of volumes under maps more closely.

2.5.16 Definition. *Let M and N be two orientable n-manifolds with volumes Ω_M and Ω_N, respectively. Then we call a C^∞ map $f: M \to N$ **volume preserving** (with respect to Ω_M and Ω_N) if $f^*\Omega_N = \Omega_M$, and we call f **orientation preserving** if $f^*(\Omega_N) \in [\Omega_M]$, and **orientation reversing** if $f^*(\Omega_N) \in [-\Omega_M]$.*

From 2.5.8, $[f^*\Omega_N]$ depends only on $[\Omega_N]$. Thus the first part of the definition depends explicitly on Ω_M and Ω_N while the last two parts depend only on the orientations $[\Omega_M]$ and $[\Omega_N]$. Furthermore, we see from 2.5.8 that if f is volume preserving with respect to Ω_M, Ω_N, then f is volume preserving with respect to $h\Omega_M$, $g\Omega_N$ iff $h = g \circ f$. It is also clear that if f is volume preserving with respect to Ω_M, Ω_N, then f is orientation preserving with respect to $[\Omega_M], [\Omega_N]$.

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2.5.17 Proposition. *Let M and N be n-manifolds with volumes Ω_M and Ω_N, respectively. Suppose $f: M \to N$ is of class C^∞. Then (i) $f^*(\Omega_N)$ is a volume iff f is a local diffeomorphism; that is, for each $m \in M$, there is a neighborhood V of m such that $f|V: V \to f(V)$ is a diffeomorphism. (ii) If M is connected, then f is a local diffeomorphism iff f is orientation preserving or orientation reversing.*

Proof. If f is a local diffeomorphism, then clearly $f^*(\Omega_N)(m) \neq 0$, by 2.4.9(ii). Conversely, if $f^*(\Omega_N)$ is a volume, then the determinant of the derivative of the local representative is not zero, and hence the derivative is an isomorphism. The result then follows by the inverse function theorem. (ii) follows at once from (i) and 2.5.7. ■

Next we consider the global analog of the determinant.

2.5.18 Definition. *Suppose M and N are orientable n-manifolds with volumes Ω_M and Ω_N, respectively. If $f: M \to N$ is of class C^∞, the unique C^∞ function $det_{(\Omega_M, \Omega_N)} f \in \mathcal{F}(M)$ such that $f^*\Omega_N = (det_{(\Omega_M, \Omega_N)} f)\Omega_M$ is called the* ***determinant*** *of f (with respect to Ω_M and Ω_N). If $f: M \to M$, we write $det_{\Omega_M} f = det_{(\Omega_M, \Omega_M)} f$.*

The basic properties of determinants given in Sect. 2.3 also hold in the global case, as follows.

2.5.19 Proposition. *In the notation of 2.5.18, f is a local diffeomorphism iff $det_{(\Omega_M, \Omega_N)} f(m) \neq 0$ for all $m \in M$.*

This follows at once from 2.5.17.

2.5.20 Proposition. *Let M be an orientable manifold with volume Ω. Then*

(i) if $f: M \to M$, $g: M \to M$ are of class C^∞, then $det_\Omega(f \circ g) = [(det_\Omega f) \circ g][det_\Omega g]$;
(ii) if $h: M \to M$ is the identity, then $det_\Omega h = 1$;
(iii) if $f: M \to M$ is a diffeomorphism, then

$$det_\Omega(f^{-1}) = 1/\left[(det_\Omega f) \circ f^{-1}\right]$$

Proof. For (i),

$$\begin{aligned} det_\Omega(f \circ g)\Omega &= (f \circ g)^*\Omega = g^* \circ f^*\Omega \\ &= g^*(det_\Omega f)\Omega = ((det_\Omega f) \circ g)\, g^*\Omega \\ &= ((det_\Omega f) \circ g)(det_\Omega g)\Omega \end{aligned}$$

Part (ii) follows since, by 2.4.8 (iii), h^* is the identity. For (iii) we have

$$det_\Omega(f \circ f^{-1}) = 1 = ((det_\Omega f) \circ f^{-1})(det_\Omega f^{-1}) \quad ■$$

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If $f: U \subset \boldsymbol{E} \to \boldsymbol{E}$, then $\det f$ is the Jacobian determinant of f [that reduces to the determinant of f if f is linear since $Df(u)=f$ if f is linear]. Then in this case, (i) above is the usual "chain rule" for Jacobian determinants. (See the proof of 2.5.5.)

2.5.21 Proposition. *Let $(M,[\Omega_M])$ and $(N,[\Omega_N])$ be oriented manifolds and $f: M \to N$ be of class C^∞. Then f is orientation preserving iff $\det_{(\Omega_M,\Omega_N)} f(m) > 0$ for all $m \in M$, and orientation reversing iff $\det_{(\Omega_M,\Omega_n)} f(m) < 0$ for all $m \in M$. Also, f is volume preserving with respect to Ω_M, Ω_N iff $\det_{(\Omega_M,\Omega_N)} f = 1$.*

This proposition follows at once from the definitions. Note that the first two assertions depend only on the orientations $[\Omega_M]$ and $[\Omega_N]$ since

$$\det_{(h\Omega_M, g\Omega_N)} f = \left(\frac{g \circ f}{h}\right) \det_{(\Omega_M,\Omega_N)} f$$

which the reader can easily check. Here $g \in \mathcal{F}(N)$, $h \in \mathcal{F}(M)$, $g(n) \neq 0$, and $h(m) \neq 0$ for all $n \in N$, $m \in M$.

Suppose that X is a vector field on $\boldsymbol{R}^n$ and $\Omega_0 = dx^1 \wedge \cdots \wedge dx^n$ is the standard volume on $\boldsymbol{R}^n$. Then $\boldsymbol{L}_X \Omega_0 = \boldsymbol{L}_X dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n + \cdots + dx^1 \wedge \cdots \wedge \boldsymbol{L}_X dx^n$ (since $\boldsymbol{L}_X$ is a derivation). But $\boldsymbol{L}_X dx^i = d\boldsymbol{L}_X x^i$ and $\boldsymbol{L}_X x^i = dx^i(X) = X^i$, the components of X. Hence

$$\boldsymbol{L}_X dx^i = dX^i = \left(\frac{\partial X^i}{\partial x^j}\right) dx^j \quad \text{and} \quad \boldsymbol{L}_X \Omega_0 = \left(\frac{\partial X^i}{\partial x^i}\right)\Omega_0$$

since $dx^i \wedge dx^i = 0$. That is, $\boldsymbol{L}_X \Omega_0 = (\operatorname{div} X)\Omega_0$ where $\operatorname{div} X$ is the usual divergence of a vector field on $\boldsymbol{R}^n$. The generalization of this is as follows.

2.5.22 Definition. *Let M be an orientable manifold with volume Ω, and X a vector field on M. Then the unique function $\operatorname{div}_\Omega X \in \mathcal{F}(M)$, such that $\boldsymbol{L}_X \Omega = (\operatorname{div}_\Omega X)\Omega$ is called the* ***divergence*** *of X. We say X is* ***incompressible*** *(with respect to Ω) iff $\operatorname{div}_\Omega X = 0$.*

2.5.23 Proposition. *Let M be an orientable manifold with volume Ω, and X a vector field on M. Then:*

(i) if $f \in \mathcal{F}(M)$ and $f(m) \neq 0$ for all $m \in M$, then

$$\operatorname{div}_{f\Omega} X = \operatorname{div}_\Omega X + \frac{\boldsymbol{L}_X f}{f}$$

(ii) for $g \in \mathcal{F}(M)$, $\operatorname{div}_\Omega gX = g \operatorname{div}_\Omega X + \boldsymbol{L}_X g$.

Proof. Since $\boldsymbol{L}_X$ is a derivation, we have

$$\boldsymbol{L}_X(f\Omega) = (\boldsymbol{L}_X f)\Omega + f\boldsymbol{L}_X \Omega$$

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As $f\Omega$ is a volume, $(div_{f\Omega}X)(f\Omega) = (\boldsymbol{L}_X f)\Omega + f(div_\Omega X)\Omega$. Then (i) follows. For (ii), we have, by 2.4.13, $\boldsymbol{L}_{gX}\Omega = g\boldsymbol{L}_X\Omega + \boldsymbol{d}g \wedge \boldsymbol{i}_X\Omega$. Now from the antiderivation property of $\boldsymbol{i}_X$, $\boldsymbol{d}g \wedge \boldsymbol{i}_X\Omega = -\boldsymbol{i}_X(\boldsymbol{d}g \wedge \Omega) + \boldsymbol{i}_X\,\boldsymbol{d}g \wedge \Omega$. But $\boldsymbol{d}g \wedge \Omega \in \Omega^{n+1}(M)$, and hence $\boldsymbol{d}g \wedge \Omega = 0$. Also, $\boldsymbol{i}_X\,\boldsymbol{d}g = \boldsymbol{L}_X g$ and so $\boldsymbol{L}_{gX}\Omega = g\boldsymbol{L}_X\Omega + (\boldsymbol{L}_X g)\Omega$. The result follows at once from this. ■

2.5.24 Proposition. *Let M be a manifold with volume Ω and X a vector field on M. Then X is incompressible (with respect to Ω) iff every flow box of X is volume preserving; that is, for the diffeomorphism $F_\lambda: U \to V, F_\lambda$ is volume preserving with respect to $\Omega|U$ and $\Omega|V$.*

Proof. If X is incompressible, $\boldsymbol{L}_X\Omega = 0$, Ω is constant along integral curves of X; $\Omega(m) = (F_\lambda)^*\Omega(m)$. Hence F_λ is volume preserving. Conversely, if $(F_\lambda)^*\Omega(m) = \Omega(m)$, then $\boldsymbol{L}_X\Omega = 0$. ■

2.5.25 Corollary. *Let M be an orientable manifold with volume Ω, and X a complete vector field with flow F on M. Then X is incompressible iff $det_\Omega F_\lambda = 1$ for all $\lambda \in \boldsymbol{R}$.*

EXERCISES

2.5A. Let $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a diffeomorphism with positive Jacobian and $f(\boldsymbol{0}) = \boldsymbol{0}$. Prove that there is a continuous curve f_t of diffeomorphisms joining f to the identity. [*Hint*: first join f to $Df(0)$ by $g_t(x) = f(tx)/t$.]

2.5B. If t is a tensor density of M, that is, $t = t' \otimes \mu$, where μ is a volume, show that

$$L_X t = (L_X t') \otimes \mu + (div_\mu X) t \otimes \mu$$

2.5C. (T. Hughes) A map $A: \boldsymbol{E} \to \boldsymbol{E}$ is said to be derived from a variational principle if there is a function $L: \boldsymbol{E} \to \boldsymbol{R}$ such that

$$\boldsymbol{d}L(x)\cdot \boldsymbol{v} = \langle A(x), \boldsymbol{v}\rangle$$

where $\langle , \rangle$ is an inner product on $\boldsymbol{E}$. Prove Vainberg's theorem: *A comes from a variational principle if and only if $DA(x)$ is a symmetric linear operator.* Do this by applying the Poincaré lemma to the one form $\boldsymbol{\alpha}(x)\cdot\boldsymbol{v} = \langle A(x), \boldsymbol{v}\rangle$.

2.6 INTEGRATION ON MANIFOLDS

The aim of this section is to define the integral of an n-form on an n-manifold M. We begin with a summary of the basic results on $\boldsymbol{R}^n$.

Suppose $f: \boldsymbol{R}^n \to \boldsymbol{R}$ is continuous and has compact support. Then $\int f dx^1 \cdots dx^n$ is defined as the Riemann integral over any rectangle containing the support of f (see Marsden [1974a, Chapter 9]).

2.6.1 Definition. *Let $U \subset \boldsymbol{R}^n$ be open and $\omega \in \Omega^n(U)$ have compact support.*

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If, relative to the standard basis of $\boldsymbol{R}^n$,

$$\omega(u)=\frac{1}{n!}\omega_{i_1\cdots i_n}(u)\,dx^{i_1}\wedge\cdots\wedge dx^{i_n}=\omega_{1\cdots n}(u)\,dx^1\wedge\cdots\wedge dx^n$$

where

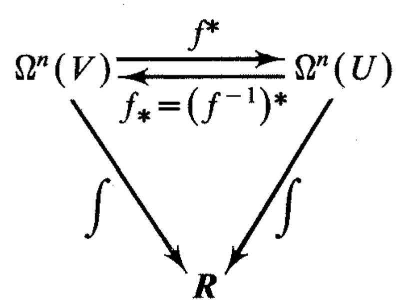
$$\omega_{i_1\cdots i_n}(u)=\omega(u)(e_{i_1},\ldots,e_{i_n})$$

we define

$$\int\omega=\int\omega_{1\cdots n}(u)\,dx^1\cdots dx^n$$

Clearly, if we regard $\omega\in\Omega^n(\boldsymbol{R}^n)$, the integral is unchanged.
The change of variables rule takes the following form.

2.6.2 Theorem. *Let* U, V *be open subsets of* $\boldsymbol{R}^n$ *and suppose* $f\colon U\to V$ *is an orientation preserving diffeomorphism. Then if* $\omega\in\Omega^n(V)$ *has compact support,* $f^*\omega\in\Omega^n(U)$ *has compact support and* $\int f^*\omega=\int\omega$, *that is, the following diagram commutes:*



Proof. If $\omega=\omega_{1\ldots n}\,dx^1\wedge\cdots\wedge dx^n$, then $f^*\omega=(\omega_{1\ldots n}\circ f)(\det_{\Omega_0}f)\Omega_0$, where $\Omega_0=dx^1\wedge\cdots\wedge dx^n$ is the standard volume on $\boldsymbol{R}^n$. Since f is a diffeomorphism, the support of $f^*\,\omega$ is compact. Then

$$\int f^*\omega=\int(\omega_{1\cdots n}\circ f)\big(\det_{\Omega_0}f\big)\,dx^1\cdots dx^n$$

As was discussed in Sect. 2.5, $\det_{\Omega_0}f>0$ is the Jacobian determinant of f. Now by covering the support of ω by a finite number of disks, we see that the usual change of variables formula applies in this case (Marsden [1974a, Chapter 9]), namely,

$$\int\omega_{1\cdots n}\,dx^1\cdots dx^n=\int(\omega_{1\cdots n}\circ f)\big(\det_{\Omega}f\big)\,dx^1\cdots dx^n$$

which implies $\int f^*\omega=\int\omega$. ■

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Suppose that (U,φ) is a chart on a manifold M, and $\omega \in \Omega^n(M)$. Then if *supp* $\omega \subset U$, we may form $\omega|U$, which has the same support. Then $\varphi_*(\omega|U)$ has compact support, and we may state the following.

2.6.3 Definition. *Let M be an orientable n-manifold with orientation Ω. Suppose $\omega \in \Omega^n(M)$ has compact support $C \subset U$, where (U,φ) is a positively oriented chart. Then we define $\int_{(\varphi)}\omega = \int \varphi_*(\omega|U)$.*

2.6.4 Proposition. *Suppose $\omega \in \Omega^n(M)$ has compact support $C \subset U \cap V$, where (U,φ), (V,ψ) are two positively oriented charts on the oriented manifold M. Then*

$$\int_{(\varphi)} \omega = \int_{(\psi)} \omega$$

Proof. By 2.6.2, $\int \varphi_*(\omega|U) = \int (\psi \circ \varphi^{-1})_* \varphi_*(\omega|U)$. Hence $\int \varphi_*(\omega|U) = \int \psi_*(\omega|U)$. [Recall that for diffeomorphisms $f_* = (f^{-1})^*$ and $(f \circ g)_* = f_* \circ g_*$.] ■

Thus we merely define $\int \omega = \int_{(\varphi)} \omega$, where (U,φ) is any positively oriented chart containing the compact support of ω (if one exists).

More generally, we can define $\int \omega$ where ω has compact support as follows.

2.6.5 Definition. *Let M be an oriented manifold and $\mathcal{A}$ an atlas of positively oriented charts. Let $P = \{(U_\alpha, \varphi_\alpha, g_\alpha)\}$ be a partition of unity subordinate to $\mathcal{A}$. Define $\omega_\alpha = g_\alpha \omega$ (so ω_α has compact support in some U_i). Then define*

$$\int_P \omega = \sum_\alpha \int \omega_\alpha$$

2.6.6 Proposition. (*i*) *The above sum contains only a finite number of non-zero terms, and hence $\int_P \omega \in \mathbf{R}$.*

(*ii*) *For any other atlas of positively oriented charts and subordinate partition of unity Q we have $\int_P \omega = \int_Q \omega$.*

The common value is denoted $\int \omega$, the **integral** of $\omega \in \Omega^n(M)$.

Proof. For any $m \in M$, there is a neighborhood U such that only a finite number of g_α are nonzero on U. By compactness of *supp* ω, a finite number of such neighborhoods cover *supp* ω. Hence only a finite number of g_α are nonzero on the union of these U. For (ii), let $P = \{(U_\alpha, \varphi_\alpha, g_\alpha)\}$ and $Q = \{(V_\beta, \psi_\beta, h_\beta)\}$ be two partitions of unity with positively oriented charts. Then

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the functions $\{g_\alpha h_\beta\}$ have $g_\alpha h_\beta(m)=0$ except for a finite number of indices (α,β), and $\Sigma_\alpha\Sigma_\beta g_\alpha h_\beta(m)=1$, for all $m\in M$. Hence, since $\Sigma_\beta h_\beta=1$,

$$\begin{aligned}\int_P \omega &= \sum_\alpha \int g_\alpha \omega \\ &= \sum_\beta \sum_\alpha \int h_\beta g_\alpha \omega \\ &= \sum_\alpha \sum_\beta \int g_\alpha h_\beta \omega = \int_Q \omega \qquad \blacksquare\end{aligned}$$

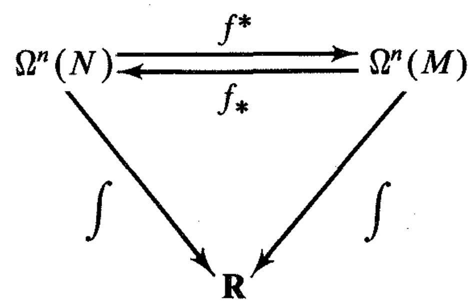
The globalization of the change of variables formula is as follows.

2.6.7 Theorem. *Suppose M and N are oriented n-manifolds and $f\colon M\to N$ is an orientation preserving diffeomorphism. If $\omega\in\Omega^n(N)$ has compact support then $f^*\omega$ has compact support and $\int\omega=\int f^*\omega$.*

Proof. First, $\operatorname{supp} f^*\omega=f^{-1}(\operatorname{supp}\omega)$, which is compact. For the second part, let $\{U_i,\varphi_i\}$ be an atlas of positively oriented charts of M and let $P=\{g_i\}$ be a subordinate partition of unity. Then $\{f(U_i),\varphi_i\circ f^{-1}\}$ is an atlas of positively oriented charts of N and $Q=\{g_i\circ f^{-1}\}$ is a partition of unity subordinate to the covering $\{f(U_i)\}$. Then

$$\begin{aligned}\int f^*\omega &= \sum_i \int g_i f^*\omega = \sum_i \int \varphi_{\alpha *}(g_i f^*\omega) \\ &= \sum_i \int \varphi_{\alpha *}\cdot(f^{-1})_*(g_i\circ f^{-1})\omega \\ &= \sum_i \int (\varphi_\alpha\circ f^{-1})_*(g_i\circ f^{-1})\omega \\ &= \int \omega \qquad \blacksquare\end{aligned}$$

As in 2.6.2, we have the following commutative diagram:



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We also can integrate functions of compact support as follows.

2.6.8 Definition. *Let M be an orientable manifold with volume Ω. Suppose $f\in\mathfrak{F}(M)$ and f has compact support. Then we define $\int_\Omega f=\int f\Omega$, the **integral** of f with respect to Ω.*

The reader can easily check that since the Riemann integral is $\boldsymbol{R}$ linear, so is the integral above.

The next theorem will show that the foregoing integral can be obtained in a unique way from a measure on M. (The reader unfamiliar with measure theory can find the necessary background in Royden [1963]. However, this will not be essential for future sections.) The integral we have described can clearly be extended to all continuous functions with compact support. Then we have the following.

2.6.9 Theorem (Riesz representation theorem). *Let M be an orientable manifold with volume Ω. Let $\mathfrak{B}$ denote the Borel sets of M, the σ algebra generated by the open (or closed, or compact) subsets of M. Then there is a unique measure μ_Ω on $\mathfrak{B}$ (and hence a completion $\bar{\mu}_\Omega$) such that for every continuous function of compact support, $\int f\,d\mu_\Omega=\int_\Omega f$.*

Proof. Existence of such a μ_Ω is proved in Royden [1963, p. 251]. For uniqueness, it is enough to consider bounded open sets (by the Hahn extension theorem). Thus, let U be open in M, and let C_U be its characteristic function. We can construct a sequence of C^∞ functions of compact support φ_n such that $\varphi_n\downarrow C_U$, pointwise. Hence from the monotone convergence theorem $\int_\Omega\varphi_n=\int\varphi_n\,d\mu_\Omega\to\int C_U\,d\mu_\Omega=\mu_\Omega(U)$. Thus, μ_Ω is unique. ■

Then one can define the space $L^p(M,\Omega)$, $p\in\boldsymbol{R}$, consisting of all measurable functions f such that $|f|^p$ is integrable. For $p\geqslant 1$, the norm $\|f\|_p=(\int|f|^p\,d\mu_\Omega)^{1/p}$ makes $L^p(M,\Omega)$ into a Banach space (functions that differ only on a set of measure zero are identified).

The behavior of these spaces under mappings can give information about the manifold. In particular, the effect under flows is of importance in statistical mechanics. In this connection we have the following.

2.6.10 Proposition. *Let M be an orientable manifold with volume Ω. Suppose X is a complete vector field on M with flow F. Then X is incompressible iff μ_Ω is F invariant, that is, $\int f\,d\mu_\Omega=\int f\circ F_\lambda\,d\mu_\Omega$ for all λ, and $f\in L^1(M,\Omega)$.*

Proof. If X is incompressible, and f is continuous with compact support, then $\int(f\circ F_\lambda)\Omega=\int f\circ F_\lambda(F_\lambda)^*\Omega=\int(F_\lambda)^*(f\Omega)=\int f\Omega$. Hence, by uniqueness in 2.6.9, we have $\int f\,d\mu_\Omega=\int(f\circ F_\lambda)\,d\mu_\Omega$ for all integrable f. Conversely, if

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$\int (f \circ F_\lambda) d\mu_\Omega = \int f d\mu_\Omega$, then taking f continuous with compact support, we see

$$\int (f \circ F_\lambda)\Omega = \int (f \circ F_\lambda) F_{\lambda *}\Omega$$

$$= \int (f \circ F_\lambda)(det_\Omega F_\lambda)\Omega$$

Thus, for every integrable f, $\int f d\mu_\Omega = \int (f\, det_\Omega F_\lambda) d\mu_\Omega$. Hence, $det_\Omega F_\lambda = 1$, which implies X is incompressible. ■

We now make a number of remarks and definitions preparatory to proving Stokes' theorem.

Let $\boldsymbol{R}^n_+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \boldsymbol{R}^n | X_n \geqslant 0\}$ denote the upper *half-space* of $\boldsymbol{R}^n$ and let $U \subset \boldsymbol{R}^n_+$ be an open set (in the topology induced on $\boldsymbol{R}^n_+$ from $\boldsymbol{R}^n$). Call $Int\ U = U \cap \{x \in \boldsymbol{R}^n | x_n > 0\}$ the *interior* of U and $\partial U = U \cap (\boldsymbol{R}^{n-1} \times \{0\})$ the *boundary* of U. We clearly have $U = Int\ U \cup \partial U$, $Int\ U$ is open in U, ∂U closed *in* U (*not* in $\boldsymbol{R}^n$), and $\partial U \cap Int\ U = \varnothing$.

Let U, V be open sets in $\boldsymbol{R}^n_+$ and $f\colon U \to V$. We shall say that f is ***smooth*** if for each point $\boldsymbol{x} \in U$ there exist open neighborhoods U_1 of x and V_1 of $f(\boldsymbol{x})$ in $\boldsymbol{R}^n$ and a smooth map $f_1\colon U_1 \to V_1$ such that $f|U \cap U_1 = f_1|U \cap U_1$. We then define $Df(\boldsymbol{x}) = Df_1(\boldsymbol{x})$. We must prove that this definition is independent of the choice of f_1, that is, we have to show that if $\phi\colon W \to \boldsymbol{R}^n$ is a smooth map with W open in $\boldsymbol{R}^n$ such that $\phi | W \cap \boldsymbol{R}^n_+ = 0$, then $D\phi(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in W \cap \boldsymbol{R}^n_+$. If $\boldsymbol{x} \in Int(W \cap \boldsymbol{R}^n_+)$, there is nothing to prove. If $\boldsymbol{x} \in \partial(W \cap \boldsymbol{R}^n_+)$, choose a sequence $\boldsymbol{x}_n \in Int(W \cap \boldsymbol{R}^n_+)$ such that $\boldsymbol{x}_n \to \boldsymbol{x}$; but then $0 = D\phi(\boldsymbol{x}_n) \to D\phi(\boldsymbol{x})$ and hence $D\phi(\boldsymbol{x}) = 0$, which proves our claim.

Let $U \subset \boldsymbol{R}^n_+$ be open, $\phi\colon U \to \boldsymbol{R}^n_+$ be a smooth map, and assume that for some $\boldsymbol{x}_0 \in Int\ U$, $\phi(\boldsymbol{x}_0) \in \partial \boldsymbol{R}^n_+$. We claim that $D\phi(\boldsymbol{x}_0)(\boldsymbol{R}^n) \subset \partial \boldsymbol{R}^n_+$. To see this, let $p_n\colon \boldsymbol{R}^n \to \boldsymbol{R}$ be the canonical projection onto the nth factor and notice that the relation

$$\phi(\boldsymbol{x}_0 + t\boldsymbol{x}) = \phi(\boldsymbol{x}_0) + D\phi(\boldsymbol{x}_0)\cdot t\boldsymbol{x} + o(t\boldsymbol{x})$$

where $lim_{t \to 0} o(t\boldsymbol{x})/t = 0$, together with the hypothesis $(p_n \circ \phi)(y) \geqslant 0$ for all $y \in U$, implies $0 \leqslant (p_n \circ \phi)(\boldsymbol{x}_0 + t\boldsymbol{x}) = 0 + (p_n \circ D\phi)(\boldsymbol{x}_0)\cdot t\boldsymbol{x} + p_n(o(t\boldsymbol{x}))$, whence for $t > 0$

$$0 \leqslant (p_n \circ D\phi)(\boldsymbol{x}_0)\cdot \boldsymbol{x} + p_n \circ \left(\frac{o(t\boldsymbol{x})}{t}\right)$$

Letting $t \to 0$, we get $(p_n \circ D\phi)(\boldsymbol{x}_0)\cdot \boldsymbol{x} \geqslant 0$ for all $\boldsymbol{x} \in \boldsymbol{R}^n$. Similarly, for $t < 0$, letting $t \to 0$, we get $(p_n \circ D\phi)(\boldsymbol{x}_0)\cdot \boldsymbol{x} \leqslant 0$ for all $\boldsymbol{x} \in \boldsymbol{R}^n$. The conclusion is

$$(D\phi)(\boldsymbol{x}_0)(\boldsymbol{R}^n) \subset \boldsymbol{R}^{n-1} \times \{0\}$$

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We now prove the following assertion:

Lemma. *Let U, V be open sets in $\boldsymbol{R}^n_+$ and $f\colon U\to V$ a diffeomorphism. Then f induces diffeomorphisms $Int\, f\colon Int\, U\to Int\, V$ and $\partial f\colon \partial U\to\partial V$.*

Proof. Assume first that $\partial U=\varnothing$, that is, that $U\cap(\boldsymbol{R}^{n-1}\times\{0\})=\varnothing$. We shall show that $\partial V=\varnothing$ and hence we take $Int\, f=f$. If $\partial V\neq\varnothing$, there exists $\boldsymbol{x}\in U$ such that $f(\boldsymbol{x})\in\partial V$ and hence by definition of smoothness in $\boldsymbol{R}^n_+$, there are open neighborhoods in $\boldsymbol{R}^n$, $U_1\subset U$, $\boldsymbol{x}\in U_1$, $V_1\subset\boldsymbol{R}^n$, $f(\boldsymbol{x})\in V_1$, and smooth maps $f_1\colon U_1\to V_1$, $g_1\colon V_1\to U_1$ such that $f|U_1=f_1$, $g_1|V\cap V_1=f^{-1}|V\cap V_1$. Let $\boldsymbol{x}_n\in U_1$, $\boldsymbol{x}_n\to\boldsymbol{x}$, $\boldsymbol{y}_n\in V_1\setminus\partial V$, and $\boldsymbol{y}_n=f(\boldsymbol{x}_n)$. We have

$$\begin{aligned} Df(\boldsymbol{x})\circ Dg_1(f(\boldsymbol{x})) &= \lim_{\boldsymbol{y}_n\to f(\boldsymbol{x})}(Df(g_1(\boldsymbol{y}_n))\circ Dg_1(\boldsymbol{y}_n)) \\ &= \lim_{\boldsymbol{y}_n\to f(\boldsymbol{x})} D(f\circ g_1)(\boldsymbol{y}_n)=id_{\boldsymbol{R}^n} \end{aligned}$$

and similarly

$$Dg_1(f(\boldsymbol{x}))\circ Df(\boldsymbol{x})=id_{\boldsymbol{R}^n}$$

so that $Df(\boldsymbol{x})^{-1}$ exists and equals $Dg_1(f(\boldsymbol{x}))$. But we saw above that $Df(\boldsymbol{x})(\boldsymbol{R}^n)\subset\boldsymbol{R}^{n-1}\times\{0\}$, which is impossible, $Df(\boldsymbol{x})$ being an isomorphism.

Assume that $\partial U\neq\varnothing$. If we assume $\partial V=\varnothing$, then, working with f^{-1} instead of f, the above argument leads to a contradiction. Hence $\partial V\neq\varnothing$. Let $\boldsymbol{x}\in Int\, U$ so that $\boldsymbol{x}$ has a neighborhood $U_1\subset U$, $U_1\cap\partial U=\varnothing$, and hence $\partial U_1=\varnothing$. Thus, by the above argument, $\partial f(U_1)=\varnothing$, and $f(U_1)$ is open in $V\setminus\partial V$. This shows that $f(Int\, U)\subset Int\, V$. Similarly, working with f^{-1}, we conclude $f(Int\, U)\supset Int\, V$ and hence $f\colon Int\, U\to Int\, V$ is a diffeomorphism. But then $f(\partial U)=\partial V$ and $f|\partial U\colon\partial U\to\partial V$ is a diffeomorphism as well. ■

Now we define a ***manifold with boundary*** exactly as in Sect. 1.4 with the following difference: if (U,ϕ) is a chart, we require that $\phi(U)\subset\boldsymbol{R}^n_+$. Let $\mathcal{A}=\{(U,\phi)\}$ be an atlas on the manifold with boundary M. Define $Int\, M=\cup_U\phi^{-1}(Int(\phi(U)))$ and $\partial M=\cup_U\phi^{-1}(\partial(\phi(U)))$ called, respectively, the ***interior*** and ***boundary*** of M. Their definition makes sense by the lemma above. $Int\, M$ is open in M and so is an n-dimensional manifold; ∂M is an $(n-1)$-dimensional manifold (possibly empty) without boundary.

If $\mathcal{A}=\{(U,\phi)\}$ is an atlas on M, then the atlas $\mathcal{B}=\{(\partial U, p_n\circ\partial\phi)\}$, $p_n\circ\partial\phi\colon\partial U\to\partial\phi(U)\subset\boldsymbol{R}^{n-1}$ defines the manifold structure on ∂M.

Summarizing, we have proved the following.

2.6.11 Proposition. *If M is an n-manifold with boundary, then its interior $Int\, M$ and its boundary ∂M are smooth manifolds without boundary of dimension n and $n-1$, respectively. Moreover, if $f\colon M\to N$ is a diffeomorphism, N being another n-manifold with boundary, then f induces, by restriction, two diffeomorphisms $Int\, f\colon Int\, M\to Int\, N$ and $\partial f\colon\partial M\to\partial N$.*

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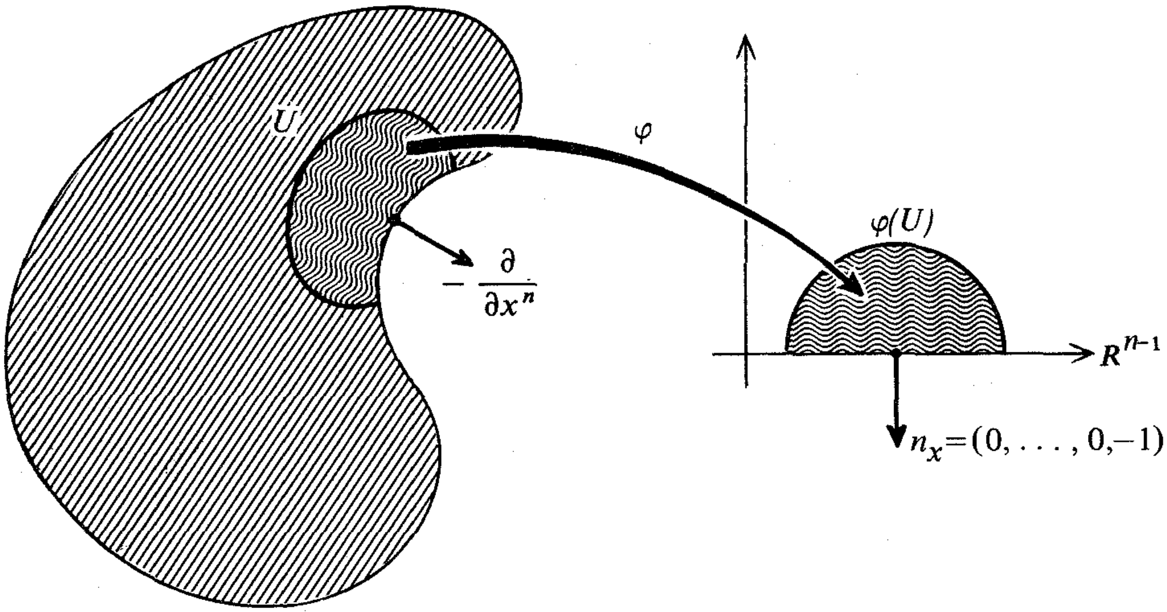

Figure 2.6-1.

Our next goal is Stokes' theorem, which deals with integration, so we have to define orientation on a manifold with boundary. A glance at the definition of orientability shows that the definition extends without difficulty to the case of manifolds with boundary. It is convenient to have in mind the following geometric interpretation of an orientation on M. An orientation on M is just a smooth choice of orientations of all the tangent spaces, "smooth" meaning that for all the charts of a certain atlas, the ***oriented charts***, the maps $D(\phi_j \circ \phi_i^{-1})(x)$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ are orientation preserving. With this picture in mind, we can define the ***boundary orientation*** of ∂M in the following way. At every $x \in \partial M$, $T_x(\partial M)$ has codimension one in $T_x(M)$ so that there are—in a chart on M intersecting ∂M—exactly two vectors perpendicular to $x_n = 0$: one points inward, the other outward. Our assertion preceding 2.6.11 assures us that a change of chart does not affect the quality of a vector being outward or inward. (See Fig. 2.6-1.)

We shall say that a basis $\{v_1, \ldots, v_{n-1}\}$ of $T_x(\partial M)$ is *positively oriented* if $\{-\partial/\partial x^n, v_1, \ldots, v_{n-1}\}$ is positively oriented in the orientation of M. This defines the *induced orientation on* ∂M.

2.6.12 Stokes' Theorem. *Let M be an oriented smooth n-manifold with boundary and $\alpha \in \Omega^{n-1}(M)$ have compact support. Let i: $\partial M \to M$ be the inclusion map so that $i^*\alpha \in \Omega^{n-1}(\partial M)$. Then*

$$\int_{\partial M} i^*\alpha = \int_M d\alpha$$

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