$T_{c(t)}f(c'(t))=0$ so that $f\circ c$ is constant on $(-\varepsilon,\varepsilon)$. Then $(f\circ c)'(t)=$ contradicting injectivity of f.)

2.3 EXTERIOR ALGEBRA

The calculus of Cartan concerns exterior differential forms, which are sections of a vector bundle of linear exterior forms on the tangent spaces of a manifold. We begin with the exterior algebra of a vector space and extend this fiberwise to a vector bundle. As with tensor fields, the most important case is the tangent bundle of a manifold, which is considered in the next section.

2.3.1 Definition. *Let* $\mathbf{E}$ *be a finite-dimensional real vector space. Let* $\Omega^0(\mathbf{E})=\mathbf{R}$. $\Omega^1(\mathbf{E})=\mathbf{E}^*$, *and, in general,* $\Omega^k(\mathbf{E})=L_a^k(\mathbf{E},\mathbf{R})$, *the vector space of skew symmetric k multilinear maps or* ***exterior k-forms*** *on* $\mathbf{E}$.

We leave as an easy exercise the fact that $\Omega^k(\mathbf{E})$ is a vector subspace of $T_k^0(\mathbf{E})$.

Recall that the permutation group on k elements, denoted S_k, consists of all bijections $\varphi:\{1,\ldots,k\}\to\{1,\ldots,k\}$ together with the structure of a group under composition. Clearly, S_k has order $k!$. Letting $(\tilde{\mathbf{R}},\times)$ denote $\mathbf{R}\backslash\{0\}$ with the multiplicative group structure, we have a homomorphism *sign*: $S_k\to(\tilde{\mathbf{R}},\times)$. That is, for $\sigma,\tau\in S_k$, $sign(\sigma\circ\tau)=(sign\,\sigma)(sign\,\tau)$. The image of *sign* is the subgroup $\{-1,1\}$, while its kernel consists of the subgroup of even permutations. One other fact we shall need is the following, which the reader can easily check: If G is a group and $g_0\in G$, the map $R_{g_0}:G\to G:g\mapsto gg_0$ is a bijection.

2.3.2 Definition. ***The alternation mapping*** $A: T_k^0(\mathbf{E})\to T_k^0(\mathbf{E})$ *(as before, we do not index the* A*) is defined by*

$$\mathbf{A}\mathbf{t}(\mathbf{e}_1,\ldots,\mathbf{e}_k)=\frac{1}{k!}\sum_{\sigma\in S_k}(sign\,\sigma)\mathbf{t}(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(k)})$$

where the sum is over all $k!$ *elements of* S_k.

2.3.3 Proposition. *A is a linear mapping onto $\Omega^k(E)$, ... identity, and $A \circ A = A$.*

Proof. Linearity of A follows at once. If $t \in \Omega^k(E)$, then

$$At(e_1,\ldots,e_k) = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sign}\sigma)t(e_{\sigma(1)},\ldots,e_{\sigma(k)})$$

$$= \frac{1}{k!}\sum_{\sigma\in S_k} t(e_1,\ldots,e_k)$$

$$= t(e_1,\ldots,e_k)$$

since $(\operatorname{sign}\sigma)^2 = 1$ and S_k has order $k!$. Second, for $t \in T_k^0(E)$ we have

$$At(e_1,\ldots,e_k) = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sign}\sigma)t(e_{\sigma(1)},\ldots,e_{\sigma(k)})$$

$$= \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sign}\sigma\tau)t(e_{\sigma\tau(1)},\ldots,e_{\sigma\tau(k)})$$

$$= (\operatorname{sign}\tau)At(e_{\tau(1)},\ldots,e_{\tau(k)})$$

since $\sigma \mapsto \sigma\tau$ is a bijection and *sign* is a homomorphism. This proves the first two assertions, and the last follows from them. ■

Then we may define the *exterior product* as follows.

2.3.4 Definition. *If $\alpha \in T_k^0(E)$ and $\beta \in T_l^0(E)$, define $\alpha\wedge\beta \in \Omega^{k+l}(E)$ by $\alpha\wedge\beta = (k+l)!/k!l!\ A(\alpha\otimes\beta)$. (Again, we do not index $\wedge$.) In particular, for $\alpha \in T_0^0(E) = R$, we put $\alpha\wedge\beta = \beta\wedge\alpha = \alpha\beta$.*

There are several possible conventions for defining the wedge product $\wedge$. The one here conforms to Spivak [1965], and Bourbaki [1971] but not to Kobayashi–Nomizu [1963] or Guillemin–Pollack [1976]. See Robbin [1974] for a lively discussion of what conventions are possible.

Our definition of $\alpha\wedge\beta$ is the one that eliminates the largest number of constants later. The reader should prove that, for exterior forms,

$$(\alpha\wedge\beta)(e_1,\ldots,e_{k+l}) = \Sigma'(\operatorname{sign}\sigma)\alpha(e_{\sigma(1)},\ldots,e_{\sigma(k)})\beta(e_{\sigma(k+1)},\ldots,e_{\sigma(k+l)})$$

where Σ' denotes the sum over all *shuffles*; that is, permutations σ of $\{1,2,\ldots,k+l\}$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$. The basic properties of the operation $\wedge$ are given in the following.

2.3.5 Proposition. *For $\alpha \in T_k^0(E)$, $\beta \in T_l^0(E)$, and $\gamma \in T_m^0(E)$, we have*

(i) $\alpha\wedge\beta = A\alpha\wedge\beta = \alpha\wedge A\beta$;
(ii) $\wedge$ *is bilinear;*

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(*iii*) $\alpha\wedge\beta=(-1)^{kl}\beta\wedge\alpha$;
(*iv*) $\alpha\wedge(\beta\wedge\gamma)=(\alpha\wedge\beta)\wedge\gamma$.

Proof. For (i), first note that if $\sigma\in S_k$ and $\sigma t(e_1,\ldots,e_k)=t(e_{\sigma(1)},\ldots,e_{\sigma(k)})$, then $A(\sigma t)=(sign\,\sigma)At$ for

$$A(\sigma t)(e_1,\ldots,e_k)=\frac{1}{k!}\sum_{\rho\in S_k}(sign\,\rho)t(e_{\rho\sigma(1)},\ldots,e_{\rho\sigma(k)})$$

$$=\frac{1}{k!}\sum_{\rho\in S_k}(sign\,\sigma)(sign\,\rho\sigma)t(e_{\rho\sigma(1)},\ldots,e_{\rho\sigma(k)})$$

$$=(sign\,\sigma)At(e_1,\ldots,e_k)$$

since $\rho\mapsto\rho\sigma$ is a bijection. Then,

$$A(A\alpha\otimes\beta)(e_1,\ldots,e_k,\ldots,e_{k+l})=A(A\alpha(e_1,\ldots,e_k)\beta(e_{k+1},\ldots,e_{k+l}))$$

$$=A\left(\frac{1}{k!}\sum_{\tau\in S_k}sign\,\tau\alpha(e_{\tau(1)},\ldots,e_{\tau(k)})\beta(e_{k+1},\ldots,e_{k+l})\right)$$

$$=A\frac{1}{k!}\sum_{\tau\in S_k}(sign\,\tau)(\tau\alpha\otimes\beta)(e_1,\ldots,e_k,\ldots,e_{k+l})$$

$$=\frac{1}{k!}\sum_{\tau\in S_k}(sign\,\tau)A(\tau\alpha\otimes\beta)(e_1,\ldots,e_{k+l})\quad\text{(linearity of } A\text{)}$$

$$=\frac{1}{k!}\sum_{\tau\in S_k}(sign\,\tau')A\tau'(\alpha\otimes\beta)(e_1,\ldots,e_{k+l})$$

where $\tau'\in S_{k+l}$,

$$\tau'(1,\ldots,k,\ldots,k+l)=(\tau(1),\ldots,\tau(k),k+1,\ldots,k+l)$$

so $sign\,\tau=sign\,\tau'$ and $\tau\alpha\otimes\beta=\tau'(\alpha\otimes\beta)$. Thus the above becomes

$$\frac{1}{k!}\sum_{\tau\in S_k}(sign\,\tau')(sign\,\tau')A(\alpha\otimes\beta)(e_1,\ldots,e_{k+l})$$

$$=A(\alpha\otimes\beta)(e_1,\ldots,e_{k+l})\frac{1}{k!}\sum_{\tau\in S_k}1$$

$$=A(\alpha\otimes\beta)(e_1,\ldots,e_{k+l})$$

Thus $A(A\alpha\otimes\beta)=A(\alpha\otimes\beta)$; that is, $(A\alpha)\wedge\beta=\alpha\wedge\beta$.

The other equality in (i) is similar.

Now (ii) is clear since $\otimes$ is bilinear and A is linear.

For (iii), let $\sigma_0 \in S_{k+l}$ be given by $\sigma_0(1, \ldots, k+l) = (k+1, \ldots, k+l, 1, \ldots, k)$. Then $\alpha \otimes \beta(e_1, \ldots, e_{k+l}) = \beta \otimes \alpha(e_{\sigma_0(1)}, \ldots, e_{\sigma_0(k+l)})$. Hence, by the proof of (i), $A(\alpha \otimes \beta) = (sign\ \sigma_0)A(\beta \otimes \alpha)$. But $sign\ \sigma_0 = (-1)^{kl}$. Finally, (iv) follows from (i). ■

2.3.6 Definition. *The direct sum of the spaces* $\Omega^k(E)$ $(k = 0, 1, 2 \ldots)$ *together with its structure as a real vector space and multiplication induced by* $\wedge$, *is called the* ***exterior algebra*** *of* **E**, *or the* ***Grassmann algebra*** *of* **E**.

Using 2.3.5 and a simple induction argument, it follows that if α_i, $i = 1, \ldots, k$ are one-forms, then

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(e_1, \ldots, e_k) = \sum_{\sigma} (sign\, \sigma) \alpha_1(e_{\sigma(1)}) \cdots \alpha_k(e_{\sigma(k)})$$

We can now find a basis for $\Omega^k(E)$.

2.3.7 Proposition. *Let* $n = \dim E$. *Then for* $k > n$, $\Omega^k(E) = \{0\}$, *while for* $0 < k \leqslant n$, $\Omega^k(E)$ *has dimension* $\binom{n}{k}$. *The exterior algebra over* **E** *has dimension* 2^n. *Indeed, if* $\hat{e} = (e_1, \ldots, e_n)$ *is an ordered basis of* **E** *and* $\hat{e}^* = (\alpha^1, \ldots, \alpha^n)$ *its dual basis, a basis of* $\Omega^k(E)$ *is*

$$\left\{ \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \,|\, 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n \right\}$$

Proof. First we show that the indicated wedge products span $\Omega^k(E)$. If $t \in \Omega^k(E)$, then from 1.7.2 we know that

$$t = t(e_{i_1}, \ldots, e_{i_k}) \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}$$

where the summation convention indicates that this should be summed over all choices of $i_1, \ldots, i_k$ between 1 and n, not just the ordered ones of the proposition. Now if the linear operator A is applied to this sum, we have, since $t \in \Omega^k(E)$,

$$t = At = t(e_{i_1}, \ldots, e_{i_k}) A(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k})$$

so that

$$t(f_1, \ldots, f_k) = t(e_{i_1}, \ldots, e_{i_k}) \frac{1}{k!} \sum_{\sigma \in S_k} (sign\, \sigma)(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k})(f_{\sigma(1)}, \ldots, f_{\sigma(k)})$$

$$= t(e_{i_1}, \ldots, e_{i_k}) \frac{1}{k!} (\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k})(f_1, \ldots, f_k)$$

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by the above remark. Therefore,

$$t = t(e_{i_1},\ldots,e_{i_k})\frac{1}{k!}\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}$$

The sum still runs over all choices of the $i_1,\ldots,i_k$ and we want only distinct, ordered ones. However, since t is skew symmetric, the coefficient $t(e_{i_1},\ldots,e_{i_k})$ is 0 if $i_1,\ldots,i_k$ are not distinct. If they are distinct and $\sigma\in S_k$, then

$$t(e_{i_1},\ldots,e_{i_k})\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k} = t(e_{\sigma(i_1)},\ldots,e_{\sigma(i_k)})\alpha^{\sigma(i_1)}\wedge\cdots\wedge\alpha^{\sigma(i_k)}$$

since both t and the wedge product change by a factor of $\operatorname{sign}\sigma$. [Use 2.3.5(iii), where α and β are one-forms.] Since there are $k!$ of these rearrangements, we are left with

$$t = \sum_{i_1<\cdots<i_k} t(e_{i_1},\ldots,e_{i_k})\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}$$

Secondly, we show

$$\{\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k} \mid i_1<\cdots<i_k\}$$

are linearly independent. Suppose that

$$\sum_{i_1<\cdots<i_k} t_{i_1\cdots i_k}\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}=0$$

For fixed $i_1',\ldots,i_k'$, let $j_{k+1}',\ldots,j_n'$ denote the complementary set of indices, $j_{k+1}'<\cdots<j_n'$. Then

$$\sum_{i_1<\cdots<i_k} t_{i_1\cdots i_k}\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}\wedge\alpha^{j_{k+1}'}\wedge\cdots\wedge\alpha^{j_n'}=0$$

However, this reduces to

$$t_{i_1'\cdots i_k'}\alpha^1\wedge\cdots\wedge\alpha^n=0$$

But $\alpha^1\wedge\cdots\wedge\alpha^n\neq 0$, as $\alpha^1\wedge\cdots\wedge\alpha^n(e_1,\ldots,e_n)=1$. Hence

$$t_{i_1'\cdots i_k'}=0$$

The proposition now follows. ■

2.3.8 Definition. *The nonzero elements of the one-dimensional space $\Omega^n(E)$ are called* ***volume elements****. If ω_1 and ω_2 are volume elements, we say ω_1 and ω_2 are* ***equivalent*** *iff there is a $c>0$ such that $\omega_1=c\omega_2$. An equivalence class of volume elements on* **E** *is called an* ***orientation*** *on* **E**.

We shall see shortly the close relationship between volume elements and determinants.

2.3.9 Proposition. *Let $\alpha_1,\ldots,\alpha_k \in E^*$. Then $\alpha_1,\ldots,\alpha_k$ are linearly dependent iff $\alpha_1\wedge\cdots\wedge\alpha_k=0$.*

Proof. If $\alpha_1,\ldots,\alpha_k$ are linearly dependent, then

$$\alpha_i=\sum_{j\neq i} c_j\alpha_j$$

for some i. Then, since $\alpha\wedge\alpha=0$, we see $\alpha_1\wedge\cdots\wedge\alpha_k=0$. Conversely, if $\alpha_1,\ldots,\alpha_k$ are linearly independent, extend to a basis $\alpha_1,\ldots,\alpha_n$. Then $\alpha_1\wedge\cdots\wedge\alpha_n\neq 0$, by 2.3.7 and hence $\alpha_1\wedge\cdots\wedge\alpha_k\neq 0$. ■

2.3.10 Proposition. *Let $\dim(E)=n$ and $\varphi\in L(E,E)$. Then there is a unique constant $\det\varphi$, called the* **determinant** *of φ, such that $\varphi^*: \Omega^n(E)\to\Omega^n(E)$, defined by $\varphi^*\omega(e_1,\ldots,e_n)=\omega(\varphi(e_1),\ldots,\varphi(e_n))$ satisfies $\varphi^*\omega=(\det\varphi)\omega$ for all $\omega\in\Omega^n(E)$.*

Proof. Clearly $\varphi^*: \Omega^n(E)\to\Omega^n(E)$ is a linear mapping. But, from 2.3.7, $\Omega^n(E)$ is one-dimensional so that if ω_0 is a basis and $\omega=c\omega_0$, $\varphi^*\omega=c\varphi^*\omega_0=b\omega$ for some constant b, clearly unique. ■

It is easy to see that this definition of determinant is the usual one (Exercise 2.3B.) However, it has the advantage of suggesting the proper global definition (Sect. 2.5), as well as making its basic properties trivial, as follows.

2.3.11 Proposition. *Let $\varphi,\psi\in L(E,E)$. Then*

(i) $\det(\varphi\circ\psi)=(\det\varphi)(\det\psi)$;
(ii) if φ is the identity, $\det\varphi=1$;
(iii) φ is an isomorphism iff $\det\varphi\neq 0$, and in this case $\det(\varphi^{-1})=(\det\varphi)^{-1}$.

Proof. For (i), $(\varphi\circ\psi)^*\omega=\det(\varphi\circ\psi)\omega$, but $(\varphi\circ\psi)^*\omega=\psi^*\circ\varphi^*\omega$ as we see from the definitions as in 1.7.17. Hence, $(\varphi\circ\psi)^*\omega=\psi^*(\det\varphi)\omega=(\det\psi)(\det\varphi)\omega$ and (i) follows. (ii) follows at once from the definition. For (iii), suppose φ is an isomorphism with inverse φ^{-1}. Then, by (i) and (iii), $1=\det(\varphi\circ\varphi^{-1})=(\det\varphi)(\det\varphi^{-1})$, and, in particular, $\det\varphi\neq 0$. Conversely, if φ is not an isomorphism there is an $e_1\neq 0$ so $\varphi(e_1)=0$ (Exercise 1.2B). Extend to a basis $e_1,e_2,\ldots,e_n$. Then for all n-forms ω, $\varphi^*\omega(e_1,\ldots,e_n)=\omega(0,\varphi(e_2),\ldots,\varphi(e_n))=0$. Hence, $\det\varphi=0$. ■

Recall from Chapter 1 that there is a unique vector space topology on $L(E,E)$ since it is finite-dimensional. One convenient norm giving this

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$$\|\varphi\| = \sup\{\|\varphi(e)\| \mid \|e\| = 1\} = \sup\left\{\frac{\|\varphi(e)\|}{\|e\|} \,\middle|\, e \neq 0\right\}$$

where $\|e\|$ is a norm on E. (See Exercise 1.2A). Hence, for any $e \in E$,

$$\|\varphi(e)\| \leqslant \|\varphi\|\,\|e\|$$

2.3.12 Proposition. *det:* $L(E,E) \to \mathbf{R}$ *is continuous.*

Proof. Note that

$$\|\omega\| = \sup\{|\omega(e_1,\ldots,e_n)| \mid \|e_1\| = \cdots = \|e_n\| = 1\}$$

$$= \sup\{|\omega(e_1,\ldots,e_n)| / \|e_1\| \cdots \|e_n\| \mid e_1,\ldots,e_n \neq 0\}$$

is a norm on $\Omega^n(E)$ and $|\omega(e_1,\ldots,e_n)| \leqslant \|\omega\|\,\|e_1\| \cdots \|e_n\|$. Then, for $\varphi, \psi \in L(E,E)$,

$$|\det\varphi - \det\psi|\,\|\omega\| = \|\varphi^*\omega - \psi^*\omega\|$$

$$= \sup\{|\omega(\varphi(e_1),\ldots,\varphi(e_n)) - \omega(\psi(e_1),\ldots,\psi(e_n))| \mid \|e_1\| = \cdots = \|e_n\| = 1\}$$

$$\leqslant \sup\{|\omega(\varphi(e_1) - \psi(e_1), \varphi(e_2),\ldots,\varphi(e_n)| + \cdots$$

$$+ |\omega(\psi(e_1), \psi(e_2),\ldots,\varphi(e_n) - \psi(e_n))| \mid \|e_1\| = \cdots = \|e_n\| = 1\}$$

$$\leqslant \|\omega\|\,\|\varphi - \psi\|\{\|\varphi\|^{n-1} + \|\varphi\|^{n-2}\|\psi\| + \cdots + \|\psi\|^{n-1}\}$$

$$\leqslant \|\omega\|\,\|\varphi - \psi\|\,(\|\varphi\| + \|\psi\|)^{n-1}$$

Consequently, $|\det\varphi - \det\psi| \leqslant \|\varphi - \psi\|(\|\varphi\| + \|\psi\|)^{n-1}$ and the result follows. ■

In 1.3.14 and 1.7.7 we saw that the isomorphisms are an open subset of $L(E,F)$. Using the determinant, we can give a simpler proof in the finite-dimensional case.

2.3.13 Proposition. *Suppose* E *and* F *are finite-dimensional and let* $GL(E,F)$ *denote those* $\varphi \in L(E,F)$ *that are isomorphisms. Then* $GL(E,F)$ *is an open subset of* $L(E,F)$.

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Proof. If $GL(E,F)=\varnothing$, the conclusion is true. If not, there is an isomorphism $\psi \in GL(E,F)$. A map φ in $L(E,F)$ is an isomorphism if and only if $\psi^{-1}\varphi$ is also. This happens precisely when $det(\psi^{-1}\varphi)\neq 0$. Therefore, $GL(E,F)$ is the inverse image of $\mathbf{R}\backslash\{0\}$ under the map taking φ to $det(\psi^{-1}\varphi)$. Since this is continuous and $\mathbf{R}\backslash\{0\}$ is open, $GL(E,F)$ is also open. ■

In order to define pull-back φ^*t or push-forward φ_*t of a general tensor t by a map φ, φ needs to be a diffeomorphism. For covariant tensors, however, pull-back makes sense if φ is merely a C^1 map. On the vector space level, this goes as follows.

2.3.14 Definition. *Let $\varphi \in L(E,F)$. For $\alpha \in T^0_k(F)$ define the* **pull-back** *of α by φ: $\varphi^*\alpha \in T^0_k(E)$ by $\varphi^*\alpha(e_1,\ldots,e_k)=\alpha(\varphi(e_1),\ldots,\varphi(e_k))$. If $\varphi \in GL(E,F)$, we denote by φ_* the* **push-forward** *map defined in 1.7.3.*

2.3.15 Proposition. *Let $\varphi \in L(E,F)$, $\psi \in L(F,G)$. Then*

(i) $\varphi^*: T^0_k(F)\to T^0_k(E)$ is linear, and $\varphi^*(\Omega^k(F))\subset\Omega^k(E)$;

(ii) $(\psi\circ\varphi)^*=\varphi^*\circ\psi^*$;

(iii) If φ is the identity, so is φ^*;

(iv) If $\varphi\in GL(E,F)$, then $\varphi^*\in GL(T^0_k(F),T^0_k(E))$, $(\varphi^*)^{-1}=(\varphi^{-1})^*$ and $\varphi^*\Omega^k(F)=\Omega^k(E)$;

(v) If $\varphi\in GL(E,F)$, then $\varphi_*\in GL(T^0_k(E),T^0_k(F))$, $(\varphi^{-1})^*=\varphi_*$, and $(\varphi_*)^{-1}=(\varphi^{-1})_*$; if $\psi\in GL(F,G)$, $(\psi\circ\varphi)_*=\psi_*\circ\varphi_*$;

(vi) If $\alpha\in\Omega^k(F)$, $\beta\in\Omega^l(F)$, then $\varphi^*(\alpha\wedge\beta)=\varphi^*\alpha\wedge\varphi^*\beta$.

Proof. It is evident that (i) follows at once from the definition. For (ii),

$$
\begin{aligned}
(\psi\circ\varphi)^*\alpha(e_1,\ldots,e_k) &= \alpha(\psi\circ\varphi(e_1),\ldots,\psi\circ\varphi(e_k)) \\
&= \psi^*\alpha(\varphi(e_1),\ldots,\varphi(e_k)) \\
&= \varphi^*\circ\psi^*\alpha(e_1,\ldots,e_k)
\end{aligned}
$$

Then (iii) is clear and (iv) follows from (ii) and (iii). For (v), $\varphi_*\beta(f_1,\ldots,f_k)=\beta(\varphi^{-1}f_1,\ldots,\varphi^{-1}f_k)=(\varphi^{-1})^*\beta(f_1,\ldots,f_k)$ and $(\varphi_*)^{-1}=$ [illegible] $=(\varphi^{-1})_*$. Finally, $\varphi^*(\alpha\wedge\beta)(e_1,\ldots,e_{k+l})=\alpha\wedge\beta(\varphi e_1,\ldots,\varphi e_{k+l})=\varphi^*\alpha\wedge\varphi^*\beta(e_1,\ldots,e_{k+l})$. ■

As in Sect. 1.7, we can consider the exterior algebra on the fibers of a vector bundle as follows.

2.3.16 Definition. *Let φ: [illegible] be a local vector bundle map that is an isomorphism on each fiber. Then define* [illegible]

2.3.17 Proposition. *If $\varphi: U \times F \to U' \times F'$ is a local vector bundle map that is an isomorphism on each fiber, then so is φ_*. Moreover, if φ is a local vector bundle isomorphism, so is φ_*.*

Proof. This is a special case of 1.7.9. ■

2.3.18 Definition. *Suppose $\pi: E \to B$ is a vector bundle. Define*

$$\omega^k(E)|A = \bigcup_{b \in A} \Omega^k(E_b)$$

where A is a subset of B and $E_b = \pi^{-1}(b)$ is the fiber over $b \in B$. Let $\omega^k(E)|B = \omega^k(E)$ and define $\omega^k(\pi): \omega^k(E) \to B$ by $\omega^k(\pi)(t) = b$ if $t \in \Omega^k(E_b)$.

2.3.19 Theorem. *Suppose $\{E|U_i, \varphi_i\}$ is a vector bundle atlas of π, where φ_i: $E|U_i \to U_i' \times F_i'$. Then $\{\omega^k(E)|U_i, \varphi_{i_*}\}$ is a vector bundle atlas of $\omega^k(\pi)$: $\omega^k(E) \to B$, where $\varphi_{i_*}: \omega^k(E)|U_i \to U_i' \times \Omega^k(F_i')$ is defined by $\varphi_{i_*}|E_b = (\varphi_i|E_b)_*$ (as in 2.3.16).*

Proof. We must verify (VBA 1) and (VBA 2) of 1.5.2: (VBA 1) is clear; for (VBA 2) let φ_i, φ_j be two charts on π, so that $\varphi_i \circ \varphi_j^{-1}$ is a local vector bundle isomorphism. (We may assume $U_i = U_j$.) But then from 2.3.15, $\varphi_{i_*} \circ \varphi_{j_*}^{-1} = (\varphi_i \circ \varphi_j^{-1})_*$, which is a local vector bundle isomorphism by 2.3.17. ■

Because of this theorem, the vector bundle structure of $\pi: E \to B$ induces naturally a vector bundle structure on $\omega^k(\pi)$: $\omega^k(E) \to B$, which is also Hausdorff, second countable, and of constant dimension. Hereafter $\omega^k(\pi)$ will denote this vector bundle.

EXERCISES

2.3A. If $k!$ is omitted in the definition of A (2.3.2), show that $\wedge$ fails to be associative.

2.3B. Show that, in terms of components, our definition of the determinant is the usual one.

2.3C. If α is a two-form and β is a one-form, show that

$$(\alpha \wedge \beta)(e_1, e_2, e_3) = \alpha(e_1, e_2)\beta(e_3) - \alpha(e_1, e_3)\beta(e_2) + \alpha(e_2, e_3)\beta(e_1)$$

2.3D. Show that if $e_1, \ldots, e_n$ is a basis of E and $\alpha^1, \ldots, \alpha^n$ is the dual basis, then $(\alpha^1 \wedge \cdots \wedge \alpha^n)(e_1, \ldots, e_n) = 1$.

2.4 CARTAN'S CALCULUS OF DIFFERENTIAL FORMS

We now specialize the exterior algebra of the preceding section to tangent bundles and develop a differential calculus that is special to this case. This is basic to the dual integral calculus of Sect. 2.6 and to the Hamiltonian mechanics of Chapter 3.