2.3.17 Proposition. *If $\varphi: U \times F \to U' \times F'$ is a local vector bundle map that is an isomorphism on each fiber, then so is φ_*. Moreover, if φ is a local vector bundle isomorphism, so is φ_*.*

Proof. This is a special case of 1.7.9. ■

2.3.18 Definition. *Suppose $\pi: E \to B$ is a vector bundle. Define*

$$\omega^k(E)|A = \bigcup_{b \in A} \Omega^k(E_b)$$

where A is a subset of B and $E_b = \pi^{-1}(b)$ is the fiber over $b \in B$. Let $\omega^k(E)|B = \omega^k(E)$ and define $\omega^k(\pi): \omega^k(E) \to B$ by $\omega^k(\pi)(t) = b$ if $t \in \Omega^k(E_b)$.

2.3.19 Theorem. *Suppose $\{E|U_i, \varphi_i\}$ is a vector bundle atlas of π, where φ_i: $E|U_i \to U_i' \times F_i'$. Then $\{\omega^k(E)|U_i, \varphi_{i_*}\}$ is a vector bundle atlas of $\omega^k(\pi)$: $\omega^k(E) \to B$, where $\varphi_{i_*}: \omega^k(E)|U_i \to U_i' \times \Omega^k(F_i')$ is defined by $\varphi_{i_*}|E_b = (\varphi_i|E_b)_*$ (as in 2.3.16).*

Proof. We must verify (VBA 1) and (VBA 2) of 1.5.2: (VBA 1) is clear; for (VBA 2) let φ_i, φ_j be two charts on π, so that $\varphi_i \circ \varphi_j^{-1}$ is a local vector bundle isomorphism. (We may assume $U_i = U_j$.) But then from 2.3.15, $\varphi_{i_*} \circ \varphi_{j_*}^{-1} = (\varphi_i \circ \varphi_j^{-1})_*$, which is a local vector bundle isomorphism by 2.3.17. ■

Because of this theorem, the vector bundle structure of $\pi: E \to B$ induces naturally a vector bundle structure on $\omega^k(\pi)$: $\omega^k(E) \to B$, which is also Hausdorff, second countable, and of constant dimension. Hereafter $\omega^k(\pi)$ will denote this vector bundle.

EXERCISES

2.3A. If $k!$ is omitted in the definition of A (2.3.2), show that $\wedge$ fails to be associative.

2.3B. Show that, in terms of components, our definition of the determinant is the usual one.

2.3C. If $\boldsymbol{\alpha}$ is a two-form and $\boldsymbol{\beta}$ is a one-form, show that

$$(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})(e_1, e_2, e_3) = \boldsymbol{\alpha}(e_1, e_2)\boldsymbol{\beta}(e_3) - \boldsymbol{\alpha}(e_1, e_3)\boldsymbol{\beta}(e_2) + \boldsymbol{\alpha}(e_2, e_3)\boldsymbol{\beta}(e_1)$$

2.3D. Show that if $e_1, \ldots, e_n$ is a basis of E and $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^n$ is the dual basis, then $(\boldsymbol{\alpha}^1 \wedge \cdots \wedge \boldsymbol{\alpha}^n)(e_1, \ldots, e_n) = 1$.

2.4 CARTAN'S CALCULUS OF DIFFERENTIAL FORMS

We now specialize the exterior algebra of the preceding section to tangent bundles and develop a differential calculus that is special to this case. This is basic to the dual integral calculus of Sect. 2.6 and to the Hamiltonian mechanics of Chapter 3.

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If $\tau_M: TM \to M$ is the tangent bundle of a manifold M, let $\omega^k(M) = \omega^k(TM)$, and $\omega_M^k = \omega^k(\tau_M)$, so $\omega_M^k: \omega^k(M) \to M$ is the vector bundle of exterior k forms on the tangent spaces of M. Also, let $\Omega^0(M) = \mathcal{F}(M)$, $\Omega^1(M) = \mathcal{T}_1^0(M)$, and $\Omega^k(M) = \Gamma^\infty(\omega_M^k), k = 2, 3, \ldots$.

2.4.1 Proposition. *Regarding $\mathcal{T}_k^0(M)$ as an $\mathcal{F}(M)$ module, $\Omega^k(M)$ is an $\mathcal{F}(M)$ submodule.*

Proof. If $t_1, t_2 \in \Omega^k(M)$ and $f \in \mathcal{F}(M)$, we must show $f \otimes t_1 + t_2 \in \Omega^k(M)$. From 1.7.19, we have $f \otimes t_1 + t_2 \in \mathcal{T}_k^0(M)$. But, by 2.3.1, $f \otimes t_1(m) + t_2(m) \in \Omega^k(T_m M)$ and the result follows. ■

2.4.2 Proposition. *If $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M), k, l = 0, 1, \ldots, n$, define $\alpha \wedge \beta: M \to \omega^{k+l}(M)$ by $(\alpha \wedge \beta)(m) = \alpha(m) \wedge \beta(m)$. Then $\alpha \wedge \beta \in \Omega^{k+l}(M)$, and $\wedge$ is bilinear and associative.*

Proof. First, $\wedge$ is bilinear and associative by 2.3.5. To show $\alpha \wedge \beta$ is of class C^∞, consider the local representative of $\alpha \wedge \beta$ in natural charts. This is a map of the form $(\alpha \wedge \beta)_\varphi = \boldsymbol{B} \circ (\alpha_\varphi \times \beta_\varphi)$, with $\alpha_\varphi, \beta_\varphi, C^\infty$ and $\boldsymbol{B} = \wedge$, which is bilinear. Thus $(\alpha \wedge \beta)_\varphi$ is C^∞ by Leibniz' rule. ■

2.4.3 Definition. *Let $\Omega(M)$ denote the direct sum of $\Omega^k(M)$, $k = 0, 1, \ldots, n$, together with its structure as an (infinite-dimensional) real vector space and with the multiplication $\wedge$ extended componentwise to $\Omega(M)$. We call $\Omega(M)$ the* **algebra of exterior differential forms** *on M. Elements of $\Omega^k(M)$ are called* **k-forms**. *In particular, elements of $\mathcal{X}^*(M)$ are called* **one-forms**.

Note that we generally regard $\Omega(M)$ as a real vector space rather than an $\mathcal{F}(M)$ module [as with $\mathcal{T}(M)$]. The reason is that $\mathcal{F}(M) = \Omega^0(M)$ is included in the direct sum, and $f \wedge \alpha = f \otimes \alpha = f\alpha$.

2.4.4 Notation. *Let (U, φ) be a chart on a manifold M with $U' = \varphi(U) \subset \boldsymbol{R}^n$. Let $\boldsymbol{e}_i$ denote the standard basis of $\boldsymbol{R}^n$ and let $\underline{e}_i(u) = T_{\varphi(u)}\varphi^{-1}(\varphi(u), \boldsymbol{e}_i)$. Similarly let $\boldsymbol{\alpha}^i$ denote the dual basis of $\boldsymbol{e}_i$ and $\underline{\alpha}^i(u) = (T_u\varphi)^*(\varphi(u), \boldsymbol{\alpha}^i)$. [Thus, for each $u \in U, \underline{e}_i(u)$ and $\underline{\alpha}^i(u)$ are dual bases of the fiber $T_u M$.] Then if $\varphi(u) = (x^1(u), \ldots, x^n(u)) \in \boldsymbol{R}^n$, we define*

$$\frac{\partial f}{\partial x^i} = L_{\underline{e}_i} f = \frac{\partial f_\varphi}{\partial y^i} \circ \varphi$$

at points $u \in U$.

With these notations, we see $dx^i(u) = \underline{\alpha}^i(u)$, for

$$dx^i(u)(\underline{e}_j(u)) = P_2 T_u x^i \circ T_{\varphi(u)}\varphi^{-1}(\varphi(u), \boldsymbol{e}_j) = P_2 T_u (x^i \circ \varphi^{-1})(\varphi(u), \boldsymbol{e}_j)$$
$$= D(x^i \circ \varphi^{-1})(\varphi(u)) \cdot \boldsymbol{e}_j = \delta_j^i$$

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$$t(u)=t^{i_1\cdots i_r}_{j_1\cdots j_s}(u)\underline{e}_{i_1}\otimes\cdots\otimes\underline{e}_{i_r}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}$$

and for each $\omega\in\Omega^k(U)$

$$\omega(u)=\sum_{i_1<\cdots<i_k}\omega_{i_1\cdots i_k}(u)\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}(u)$$

where

$$t^{i_1\cdots i_r}_{j_1\cdots j_s}=t\left(dx^{i_1},\ldots,dx^{i_r},\underline{e}_{j_1},\ldots,\underline{e}_{j_s}\right)$$

and

$$\omega_{i_1\cdots i_k}=\omega\left(\underline{e}_{i_1},\ldots,\underline{e}_{i_k}\right)$$

The extension of d to $\Omega^k(M)$ is given by the following.

2.4.5 Theorem. *Let M be a manifold. Then there is a unique family of mappings $\mathbf{d}^k(U)$: $\Omega^k(U)\to\Omega^{k+1}(U)$ ($k=0,1,2,\ldots,n$, and U is open in M), which we merely denote by* **d**, *called the* ***exterior derivative*** *on M, such that*

(*i*) **d** *is a* $\wedge$ ***antiderivation***. *That is,* **d** *is* $\mathbf{R}$ *linear and for* $\alpha\in\Omega^k(U)$, $\beta\in\Omega^l(U)$,

$$\mathbf{d}(\alpha\wedge\beta)=\mathbf{d}\alpha\wedge\beta+(-1)^k\alpha\wedge\mathbf{d}\beta$$

(*ii*) *If* $f\in\mathfrak{F}(U)$, $\mathbf{d}f=df$ (*as defined in* 2.2.1);

(*iii*) $\mathbf{d}\circ\mathbf{d}=\mathbf{0}$ (*that is,* $\mathbf{d}^{k+1}(U)\circ\mathbf{d}^k(U)=\mathbf{0}$);

(*iv*) **d** *is* ***natural with respect to restrictions***; *that is, if* $U\subset V\subset M$ *are open and* $\alpha\in\Omega^k(V)$, *then* $\mathbf{d}(\alpha|U)=(\mathbf{d}\alpha)|U$, *or the following diagram commutes:*

$$\begin{array}{ccc}\Omega^k(V) & \xrightarrow{\ |U\ } & \Omega^k(U)\\ \mathbf{d}\downarrow & & \downarrow\mathbf{d}\\ \Omega^{k+1}(V) & \xrightarrow[\ |U\ ]{} & \Omega^{k+1}(U)\end{array}$$

As in Sect. 2.2, condition (iv) means that $\boldsymbol{d}$ is a local operator.

ISBN 0-8053-0102-X

Proof. We first consider $F^*(\psi\wedge\omega)$ when ψ is a function. Then

$$\begin{aligned} F^*(\psi\omega)(m) &= (T_mF)^*\circ\psi\omega\circ F(m)\\ &= (T_mF)^*\circ\left[(\psi\circ F)\cdot(\omega\circ F)\right](m)\\ &= \psi(F(m))F^*\omega(m) \end{aligned}$$

or $F^*(\psi\wedge\omega)=F^*\psi\wedge F^*\omega$, as $F^*\psi=\psi\circ F$ if $\psi\in\Omega^0(N)$. Then (i) follows immediately from 2.3.15(vi). For (ii) we shall show in fact that if $m\in M$, there is a neighborhood U of $m\in M$ such that $\mathbf{d}(F^*\omega|U)=(F^*\mathbf{d}\omega)|U$, which is sufficient, as F^* and $\mathbf{d}$ are both natural with respect to restriction. Let (V,φ) be a local chart at $F(m)$ and U a neighborhood of $m\in M$ with $F(U)\subset V$. Then for $\omega\in\Omega^k(V)$, we can write

$$\omega=\omega_{i_1\cdots i_k}\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$d\omega=\partial_{i_0}\omega_{i_1\cdots i_k}\,dx^{i_0}\wedge\cdots\wedge dx^{i_k},\qquad \partial_{i_0}=\frac{\partial}{\partial x^{i_0}}$$

and by (i) above

$$F^*\omega|U=(F^*\omega_{i_1\cdots i_k})F^*\,dx^{i_1}\wedge\cdots\wedge F^*\,dx^{i_k}$$

But if $\psi\in\Omega^0(N)$, $d(F^*\psi)=F^*\,d\psi$ by the composite mapping theorem, so

$$\begin{aligned} \mathbf{d}(F^*\omega|U) &= F^*(d\omega_{i_1\cdots i_k})\wedge F^*\,dx^{i_1}\wedge\cdots\wedge F^*\,dx^{i_k}\\ &= F^*(\mathbf{d}\omega)|U \end{aligned}$$

by (i) above. ■

2.4.10 Corollary. *The operator* $\mathbf{d}$ *is natural with respect to diffeomorphisms. That is, if* $F\colon M\to N$ *is a diffeomorphism, then* $F_*\mathbf{d}\omega=\mathbf{d}F_*\omega$, *or the following diagram commutes:*

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\;F_*\;} & \Omega^k(N)\\ \Big\downarrow \mathbf{d} & & \Big\downarrow \mathbf{d}\\ \Omega^{k+1}(M) & \xrightarrow[\;F_*\;]{} & \Omega^{k+1}(N) \end{array}$$

Proof. With F_* defined as $F_*=(F)^0_k$, we see that $F_*=(F^{-1})^*$. The result then follows from 2.4.9(ii). ■

0-8053-0102-X

The next few propositions give some important relations between the Lie derivative and the exterior derivative.

2.4.11 Theorem. *Let $X \in \mathfrak{X}(M)$. Then $\mathbf{d}$ is natural with respect to $\mathbf{L}_X$. That is, for $\omega \in \Omega^k(M)$ we have $\mathbf{L}_X\omega \in \Omega^k(M)$ and $\mathbf{d}\mathbf{L}_X\omega = \mathbf{L}_X\,\mathbf{d}\omega$, or the following diagram commutes:*

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\ \mathbf{L}_X\ } & \Omega^k(M) \\ \mathbf{d}\downarrow & & \downarrow\mathbf{d} \\ \Omega^{k+1}(M) & \xrightarrow[\ \mathbf{L}_X\ ]{} & \Omega^{k+1}(M) \end{array}$$

Proof. If $\alpha^1,\ldots,\alpha^k \in \Omega^1(M)$ we have

$$\mathbf{L}_X(\alpha^1 \wedge \cdots \wedge \alpha^k) = \mathbf{L}_X\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^k + \cdots + \alpha^1 \wedge \cdots \wedge \mathbf{L}_X\alpha^k$$

This follows from the fact that $\mathbf{L}_X$ is $\mathbf{R}$ linear and is a tensor derivation. Since locally $\omega \in \Omega^k(M)$ is a linear combination of such products, it readily follows that $\mathbf{L}_X\omega \in \Omega^k(M)$. For the second part, let (U,a,F) be a flow box at $m \in M$ so that from 2.2.20,

$$\mathbf{L}_X\omega(m) = \frac{d}{d\lambda}(F_\lambda^*\omega)(m)\Big|_{\lambda=0}$$

But from 2.4.10 we have $F_\lambda^*\,\mathbf{d}\omega = \mathbf{d}(F_\lambda^*\omega)$. Then, since $\mathbf{d}$ is $\mathbf{R}$ linear, commutes with $d/d\lambda$ and so $\mathbf{d}\mathbf{L}_X\omega = \mathbf{L}_X\,\mathbf{d}\omega$. ■

The foregoing proof can also be carried out in terms of local representatives.

2.4.12 Definition. *Let M be a manifold, $X \in \mathfrak{X}(M)$, and $\omega \in \Omega^{k+1}(M)$. Th define $\mathbf{i}_X\omega \in \mathfrak{T}^0_k(M)$ by*

$$\mathbf{i}_X\omega(X_1,\ldots,X_k) = \omega(X, X_1,\ldots,X_k)$$

If $\omega \in \Omega^0(M)$, we put $\mathbf{i}_X\omega = \mathbf{0}$. We call $\mathbf{i}_X\omega$ the **inner product** *of X and ω.*

2.4.13 Theorem. *We have $\mathbf{i}_X\colon \Omega^k(M) \to \Omega^{k-1}(M)$, $k = 1,\ldots,n$, and, for $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $f \in \Omega^0(M)$,*

(i) *$\mathbf{i}_X$ is a $\wedge$ antiderivation. That is, $\mathbf{i}_X$ is $\mathbf{R}$ linear and $\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X\beta)$;*
(ii) *$\mathbf{i}_{fX}\alpha = f\mathbf{i}_X\alpha$;*
(iii) *$\mathbf{i}_X\,\mathbf{d}f = \mathbf{L}_X f$;*
(iv) *$\mathbf{L}_X\alpha = \mathbf{i}_X\,\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_X\alpha$;*
(v) *$\mathbf{L}_{fX}\alpha = f\mathbf{L}_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha$.*

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Proof. That $i_X\alpha \in \Omega^{k-1}(M)$ follows at once from 2.2.8. For (i), $\boldsymbol{R}$ linearity is clear. For the second part of (i)

$$i_X(\alpha\wedge\beta)(X_2, X_3, \ldots, X_{k+l}) = (\alpha\wedge\beta)(X, X_2, \ldots, X_{k+l})$$

and

$$i_X\alpha\wedge\beta + (-1)^k\alpha\wedge i_X\beta = \frac{(k+l-1)!}{(k-1)!\,l!}A(i_X\alpha\otimes\beta) + (-1)^k\frac{(k+l-1)!}{k!\,(l-1)!}A(\alpha\otimes i_X\beta)$$

But the sum over all permutations in the last term can be replaced by the sum over $\sigma\sigma_0$, where σ_0 is the permutation $(2, 3, \ldots, k+1, 1, k+2, \ldots, k+l) \mapsto (1, 2, 3, \ldots, k+l)$ whose sign is $(-1)^k$. Hence (i) follows. For (ii), we merely note α_X is linear, and (iii) is just the definition of $L_X f$.

For (iv) we proceed by induction on k. First note that for $k=0$, (iv) reduces to (iii). Now assume that (iv) holds for k. Then a $k+1$ form may be written as $\Sigma df_i\wedge\omega_i$, where ω_i is a k form, in some neighborhood of $m\in M$. But $L_X(df\wedge\omega) = L_X\, df\wedge\omega + df\wedge L_X\omega$ and

$$\begin{aligned} i_X d(df\wedge\omega) + d i_X(df\wedge\omega) &= -i_X(df\wedge d\omega) + d(i_X df\wedge\omega - df\wedge i_X\omega) \\ &= -i_X df\wedge d\omega + df\wedge i_X d\omega + d i_X df\wedge\omega \\ &\quad + i_X df\wedge d\omega + df\wedge d i_X\omega \\ &= df\wedge L_X\omega + dL_X f\wedge\omega \end{aligned}$$

by our inductive assumption and (iii). Since $dL_X f = L_X df$, the result follows.

Finally for (v) we have

$$\begin{aligned} L_{fX}\omega = i_{fX}d\omega + d i_{fX}\omega &= f i_X d\omega + d(f i_X\omega) \\ &= f i_X d\omega + df\wedge i_X\omega + f d i_X\omega \\ &= f L_X\omega + df\wedge i_X\omega \quad ■ \end{aligned}$$

The behavior of inner products under diffeomorphisms is given by the following.

2.4.14 Proposition. *Let M and N be manifolds and $f: M\to N$ a diffeomorphism. Then, if $\omega\in\Omega^k(N)$ and $X\in\mathfrak{X}(N)$, we have*

$$i_{f^*X} f^*\omega = f^* i_X\omega$$

ISBN 0-80530102-X

that is, inner products are natural with respect to diffeomorphisms; that is, the following diagram commutes:

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{f^*} & \Omega^k(M) \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_{f^*X}} \\ \Omega^{k-1}(N) & \xrightarrow[f^*]{} & \Omega^{k-1}(M) \end{array}$$

Similarly for $Y \in \mathfrak{X}(M)$ we have the following commutative diagram:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{f_*} & \Omega^k(N) \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle i_{f_*Y}} \\ \Omega^{k-1}(M) & \xrightarrow[f_*]{} & \Omega^{k-1}(N) \end{array}$$

Proof. Let $v_1,\ldots,v_{k-1} \in T_m(M)$ and $n=f(m)$. Then by 2.4.12 and 2.4.7

$$\begin{aligned} & \boldsymbol{i}_{f^*X} f^*\omega(m)\cdot(v_1,\ldots,v_{k-1}) \\ &= f^*\omega(m)\cdot(f^*X(m), v_1,\ldots,v_{k-1}) \\ &= f^*\omega(m)\cdot(Tf^{-1}\circ X(n), v_1,\ldots,v_{k-1}) \\ &= \omega(n)\cdot(Tf\circ Tf^{-1}X(n), Tfv_1,\ldots,Tfv_{k-1}) \\ &= \boldsymbol{i}_X\omega(n)\cdot(Tfv_1,\ldots,Tfv_{k-1}) \\ &= f^*\boldsymbol{i}_X\omega(m)\cdot(v_1,\ldots,v_{k-1}) \quad \blacksquare \end{aligned}$$

The next proposition expresses $\boldsymbol{d}$ in terms of the Lie derivative (Palais [1963]).

2.4.15 Proposition. *Let $X_i \in \mathfrak{X}(M)$, $i=0,\ldots,k$, and $\omega \in \Omega^k(M)$. Then we have*

(i) $$(\boldsymbol{L}_{X_0}\omega)(X_1,\ldots,X_k) = \boldsymbol{L}_{X_0}(\omega(X_1,\ldots,X_k)) - \sum_{i=1}^{k} \omega(X_1,\ldots,\boldsymbol{L}_{X_0}X_i,\ldots,X_k)$$

(ii) $$\boldsymbol{d}\omega(X_0,X_1,\ldots,X_k) = \sum_{i=0}^{k} (-1)^i \boldsymbol{L}_{X_i}(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)) + \sum_{0\leq i<j\leq k} (-1)^{i+j}\omega(\boldsymbol{L}_{X_i}(X_j), X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k)$$

where $\hat{X}_i$ denotes that X_i is deleted.

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Proof. Part (i) is exactly condition (DO 4) following 2.2.17. For (ii) we proceed by induction. For $k=0$, it is merely $d\omega(X_0)=L_{X_0}\omega$. Assume the formula for $k-1$. Then if $\omega\in\Omega^k(M)$, we have, by 2.4.13(iv),

$$\begin{aligned} d\omega(X_0,X_1,\ldots,X_k) &= (i_{X_0}\,d\omega)(X_1,\ldots,X_k)\\ &= (L_{X_0}\omega)(X_1,\ldots,X_k)-(d(i_{X_0}\omega))(X_1,\ldots,X_k)\\ &= L_{X_0}(\omega(X_1,\ldots,X_k))\\ &\quad -\sum_1^k \omega(X_1,\ldots,L_{X_0}X_i,\ldots,X_k)\\ &\quad -(di_{X_0}\omega)(X_1,\ldots,X_k)\qquad \text{(by (i))} \end{aligned}$$

But $i_{X_0}\omega\in\Omega^{k-1}(M)$ and we may apply the induction assumption. This gives, after a simple permutation and 2.4.12,

$$\begin{aligned} (d(i_{X_0}\omega))(X_1,\ldots,X_k) &= \sum_{i=1}^k(-1)^{i-1}L_{X_i}(\omega(X_0,X_1,\ldots,\hat{X}_i,\ldots,X_k))\\ &\quad -\sum_{1\leqslant i<j\leqslant k}(-1)^{i+j}\omega(L_{X_i}X_j,X_0,X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k) \end{aligned}$$

Substituting this into the above easily yields the result. ■

2.4.16 Definition. *We call $\omega\in\Omega^k(M)$* ***closed*** *if $d\omega=0$, and* ***exact*** *if there is an $\alpha\in\Omega^{k-1}(M)$ such that $\omega=d\alpha$.*

2.4.17 Theorem. (*i*) *Every exact form is closed.*

(*ii*) (***Poincaré lemma***). *If ω is closed, then for each $m\in M$, there is a neighborhood U of m for which $\omega|U\in\Omega^k(U)$ is exact.*

Proof. Part (i) is clear since $\boldsymbol{d}\circ\boldsymbol{d}=0$. Using a local chart and 2.4.9(ii) together with 2.4.5(iv), it is sufficient to consider the case $\omega\in\Omega^k(U)$, $U\subset\boldsymbol{E}$ a disk about $\boldsymbol{0}\in\boldsymbol{E}$, to prove (ii). On U we construct an $\boldsymbol{R}$ linear mapping $\boldsymbol{H}$: $\Omega^k(U)\to\Omega^{k-1}(U)$ such that $\boldsymbol{d}\circ\boldsymbol{H}+\boldsymbol{H}\circ\boldsymbol{d}$ is the identity on $\Omega^k(U)$. This will give the result, for $d\omega=0$ implies $\boldsymbol{d}(\boldsymbol{H}\omega)=\omega$.

For $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k\in\boldsymbol{E}$ define

$$H\omega(u)(e_1,\ldots,e_{k-1})=\int_0^1 t^{k-1}\ldots$$

0-8053-0102-X

Then, by 2.4.6,

$$dH\omega(u)\cdot(e_1,\ldots,e_k)=\sum_{i=1}^{k}(-1)^{i+1}DH\omega(u)\cdot e_i(e_1,\ldots,\hat{e}_i,\ldots,e_k)$$

$$=\sum_{i=1}^{k}(-1)^{i+1}\int_0^1 t^{k-1}\omega(tu)(e_i,e_1,\ldots,\hat{e}_i,\ldots,e_k)\,dt$$

$$+\sum_{i=1}^{k}(-1)^{i+1}\int_0^1 t^{k}D\omega(tu)\cdot e_i(u,e_1,\ldots,\hat{e}_i,\ldots,e_k)\,dt$$

(The interchange of D and $\int$ is permissible, as ω is smooth and bounded over $t\in[0,1]$.) However, we also have, by 2.4.6,

$$H\,d\omega(u)\cdot(e_1,\ldots,e_k)=\int_0^1 t^k\,d\omega(tu)(u,e_1,\ldots,e_k)\,dt$$

$$=\int_0^1 t^kD\omega(tu)\cdot u(e_1,\ldots,e_k)\,dt$$

$$+\sum_{i=1}^{k}(-1)^{i}\int_0^1 t^kD\omega(tu)\cdot e_i(u,e_1,\ldots,\hat{e}_i,\ldots,e_k)\,dt$$

Hence

$$\left[dH\omega(u)+H\,d\omega(u)\right](e_1,\ldots,e_k)=\int_0^1 kt^{k-1}\omega(tu)\cdot(e_1,\ldots,e_k)\,dt$$

$$+\int_0^1 t^kD\omega(tu)\cdot u(e_1,\ldots,e_k)\,dt$$

$$=\int_0^1\frac{d}{dt}\left[t^k\omega(tu)\cdot(e_1,\ldots,e_k)\right]dt$$

$$=\omega(u)\cdot(e_1,\ldots,e_k)$$

which proves the assertion. ■

There is another proof of the Poincaré lemma that is useful to understand. This proof will help the reader master the proof of Darboux' theorem in Sect. 3.2, and is similar in spirit to the proof of Frobenius' theorem (2.2.26).

Alternative Proof of the Poincaré Lemma. We again let U be a ball about $\mathbf{0}$ in $\boldsymbol{E}$. Let, for $t>0$, $F_t(u)=tu$. Thus F_t is a diffeomorphism and, starting at

ISBN 0-8053-0102-X

$t=1$, is generated by the time-dependent vector field

$$X_t(u)=u/t$$

that is, $F_1(u)=u$ and $dF_t(u)/dt=X_t(F_t(u))$. Therefore, since ω is closed,

$$\begin{aligned}\frac{d}{dt}F_t^*\omega &= F_t^*\mathbf{L}_{X_t}\omega\\ &= F_t^*(\mathbf{d}\mathbf{i}_{X_t}\omega)\\ &= \mathbf{d}(F_t^*\mathbf{i}_{X_t}\omega)\end{aligned}$$

For $0<t_0\leqslant 1$, we get

$$\omega - F_{t_0}^*\omega = \mathbf{d}\int_{t_0}^1 F_t^*\mathbf{i}_{X_t}\omega\, dt$$

Letting $t_0\to 0$, we get $\omega=\mathbf{d}\beta$, where

$$\beta=\int_0^1 F_t^*\mathbf{i}_{X_t}\omega\, dt$$

Explicitly,

$$\beta_u(e_1,\ldots,e_{k-1})=\int_0^1 t^{k-1}\omega_{tu}(u,e_1,\ldots,e_{k-1})\,dt$$

(Note that this β agrees with that in the previous proof.) ■

See Exercise 2.4E for a relative Poincaré lemma.

It is not true that closed forms are always exact (for example, on a sphere). In fact, the quotient groups of closed forms by exact forms (called the de Rham cohomology groups of M) shed light on the manifold topology. A discussion may be found in Flanders [1963], Singer and Thorpe [1967], and in de Rham [1955].

In differential geometry the use of vector valued forms is important; that is, one replaces multilinear maps into $\mathbf{R}$ by multilinear maps into a vector space V. One can utilize the exterior calculus by taking the components of the form. For applications to geometry, see Kobayashi–Nomizu [1963], Chern [1972], or Spivak [1974].

The following table summarizes some of the important algebraic identities involving differential forms that have been obtained.

ISBN 0-8053-0102-X