is a smooth bundle map by Exercise 5.7. The fact that this is a functor is the content of parts (b) and (c) of Lemma 3.5.

Problems

5-1. If E is a vector bundle over a topological space M, show that the projection map $\pi\colon E \to M$ is a homotopy equivalence.

5-2. Prove that the space E constructed in Example 5.2, together with the projection $\pi\colon E \to \mathbb{S}^1$, is a smooth rank-1 vector bundle over $\mathbb{S}^1$, and show that it is nontrivial.

5-3. Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Suppose $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M, and for each $\alpha \in A$ we are given a smooth local trivialization $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ of E. For each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, let $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k,\mathbb{R})$ be the transition function defined by (5.3). Show that the following identity is satisfied for all $\alpha, \beta, \gamma \in A$:

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \qquad p \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{5.6}$$

(Here juxtaposition of matrices represents matrix multiplication.)

5-4. Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Suppose for each $\alpha, \beta \in A$ we are given a smooth map $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k,\mathbb{R})$ such that (5.6) is satisfied for all $\alpha, \beta, \gamma \in A$. Show that there is a smooth rank-k vector bundle $E \to M$ with smooth local trivializations $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose transition functions are the given maps $\tau_{\alpha\beta}$. [Hint: Define an appropriate equivalence relation on $\coprod_{\alpha\in A}(U_\alpha \times \mathbb{R}^k)$, and use the bundle construction lemma.]

5-5. Let $\pi\colon E \to M$ and $\widetilde{\pi}\colon \widetilde{E} \to M$ be two smooth rank-k vector bundles over a smooth manifold M. Suppose $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M such that both E and $\widetilde{E}$ admit smooth local trivializations over each U_α. Let $\{\tau_{\alpha\beta}\}$ and $\{\widetilde{\tau}_{\alpha\beta}\}$ denote the transition functions determined by the given local trivializations of E and $\widetilde{E}$, respectively. Show that E and $\widetilde{E}$ are smoothly isomorphic over M if and only if for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha\colon U_\alpha \to \mathrm{GL}(k,\mathbb{R})$ such that

$$\widetilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)^{-1}\tau_{\alpha\beta}(p)\sigma_\beta(p), \qquad p \in U_\alpha \cap U_\beta.$$

5-6. Let $U = \mathbb{S}^1 \smallsetminus \{1\}$ and $V = \mathbb{S}^1 \smallsetminus \{-1\}$, and define $\tau\colon U \cap V \to \mathrm{GL}(1,\mathbb{R})$ by

$$\tau(z) = \begin{cases} (1), & \operatorname{Im} z > 0, \\ (-1), & \operatorname{Im} z < 0. \end{cases}$$

By the result of Problem 5-4, there is a smooth real line bundle $F \to \mathbb{S}^1$ that is trivial over U and V, and has τ as transition function. Show that F is smoothly isomorphic to the Möbius bundle of Example 5.2.

5-7. Compute the transition function for $T\mathbb{S}^2$ associated with the two local trivializations determined by stereographic coordinates.

5-8. Let $\pi\colon E \to M$ be a smooth vector bundle of rank k, and suppose $\sigma_1, \dots, \sigma_m$ are independent smooth local sections over an open subset $U \subset M$. Show that for each $p \in U$ there are smooth sections $\sigma_{m+1}, \dots, \sigma_k$ defined on some neighborhood V of p such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.

5-9. Suppose E and E' are vector bundles over a smooth manifold M and $F\colon E \to E'$ is a bijective bundle map over M. Show that F is a bundle isomorphism.

5-10. Consider the following vector fields on $\mathbb{R}^4$:

$$X_1 = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^4},$$
$$X_2 = -x^3\frac{\partial}{\partial x^1} - x^4\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3} + x^2\frac{\partial}{\partial x^4},$$
$$X_3 = -x^4\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3} + x^1\frac{\partial}{\partial x^4}.$$

Show that there are smooth vector fields V_1, V_2, V_3 on $\mathbb{S}^3$ such that V_j is ι-related to X_j for $j = 1, 2, 3$, where $\iota\colon \mathbb{S}^3 \hookrightarrow \mathbb{R}^4$ is inclusion. Conclude that $\mathbb{S}^3$ is parallelizable.

5-11. Let V be a finite-dimensional vector space, and let $G_k(V)$ be the Grassmannian of k-dimensional subspaces of V. Let T be the disjoint union of all these k-dimensional subspaces:

$$T = \coprod_{S \in G_k(V)} S;$$

and let $\pi\colon T \to G_k(V)$ be the natural map sending each point $x \in S$ to S. Show that T has a unique smooth manifold structure making it into a smooth rank-k vector bundle over $G_k(V)$, with π as projection, and with the vector space structure on each fiber inherited from V. [Remark: T is sometimes called the *tautological vector bundle* over $G_k(V)$, because the fiber over each point $S \in G_k(V)$ is S itself.]

5-12. Show that the tautological vector bundle over $G_1(\mathbb{R}^2)$ is isomorphic to the Möbius bundle. (See Problems 5-2, 5-6, and 5-11.)

5-13. Let V_0 be the category whose objects are finite-dimensional real vector spaces and whose morphisms are linear isomorphisms. If $\mathcal{F}$ is a covariant functor from V_0 to itself, for each finite-dimensional vector