quotient map of the set $\{(x, y) \in \mathbb{R}^2 : |y| \le 1\}$. (It is a smooth 2-manifold with boundary.) Show that neither E nor M is orientable.

13-13. Let E be as in Problem 13-12. Show that the orientation covering of E is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

13-14. Let $U \subset \mathbb{R}^3$ be the open subset $\left\{(x, y, z) : \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 < 1\right\}$ (the solid torus of revolution bounded by the doughnut surface of Example 8.13). Define a map $F \colon \mathbb{R}^2 \to U$ by

$$F(u, v) = \Big(\cos 2\pi u(2 + \tanh v \cos \pi u), \\ \sin 2\pi u(2 + \tanh v \cos \pi u), \tanh v \sin \pi u\Big).$$

(a) Show that F descends to a smooth embedding of E into U, where E is the total space of the Möbius bundle of Problem 9-18.
(b) Let S be the image of F. Show that S is a closed embedded submanifold of U.
(c) Show that there is no normal vector field along S.
(d) Show that S has no global defining function in U.

that the action is orientation-preserving if for each $\gamma \in \Gamma$, the diffeomorphism $x \mapsto \gamma \cdot x$ is orientation-preserving. Show that M/Γ is orientable if and only if the action of Γ is orientation-preserving.

13-5. Let $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd.

13-6. Prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.

13-7. If ω is a symplectic form on a $2n$-manifold, show that $\omega \wedge \cdots \wedge \omega$ (the n-fold wedge product of ω with itself) is a nonvanishing $2n$-form on M, and thus every symplectic manifold is orientable.

13-8. Suppose M is an oriented Riemannian manifold, and $S \subset M$ is an oriented hypersurface (with or without boundary). Show that there is a unique smooth unit normal vector field along S that determines the given orientation of S.

13-9. Suppose M is a smooth orientable Riemannian manifold and $S \subset M$ is an immersed or embedded submanifold.

(a) If S has trivial normal bundle (see page 282), show that S is orientable.

(b) If S is an orientable hypersurface, show that S has trivial normal bundle.

13-10. Let M be a connected, nonorientable smooth manifold, and let $\widehat{\pi}\colon \widehat{M} \to M$ be its orientation covering.

(a) If $\widetilde{M}$ is an orientable smooth manifold and $\pi\colon \widetilde{M} \to M$ is a smooth covering map, show that there exists a smooth map $\varphi\colon \widetilde{M} \to \widehat{M}$ such that $\widehat{\pi} \circ \varphi = \pi$. [Hint: First define a smooth map $\widetilde{\varphi}\colon \widetilde{M} \to \Lambda^n_* M$ by setting $\widetilde{\varphi}(p) = \sigma^* \Omega_p$ locally, where Ω is an orientation form for $\widetilde{M}$ and σ is a suitable local section of π.]

(b) Uniqueness of the Orientation Covering: If $\pi\colon \widetilde{M} \to M$ is as above and in addition π is a two-sheeted covering, show that φ is a diffeomorphism.

13-11. Suppose S is an oriented embedded 2-manifold with boundary in $\mathbb{R}^3$, and let $C = \partial S$ with the induced orientation. By Problem 13-8, there is a unique smooth unit normal vector field N on S that determines the orientation. Let T be the oriented unit tangent vector field on C, and let V be the unique unit vector field tangent to S along C that is orthogonal to T and inward-pointing. Show that (T_p, V_p, N_p) is an oriented orthonormal basis for $\mathbb{R}^3$ at each $p \in C$.

13-12. Let E be the total space of the Möbius bundle, which is the quotient of $\mathbb{R}^2$ by the $\mathbb{Z}$-action $n \cdot (x,y) = \big(x+n, (-1)^n y\big)$ (see Problem 9-18). The *Möbius band* is the subset $M \subset E$ that is the image under the

Proposition 13.26. *Suppose M is any Riemannian manifold with boundary. There is a unique smooth outward-pointing unit normal vector field N along ∂M.*

Proof. First we prove uniqueness. At any point $p \in \partial M$, the vector space $(T_p\partial M)^\perp \subset T_pM$ is 1-dimensional, so there are exactly two unit vectors at p that are normal to ∂M. Since any unit normal vector N is obviously transverse to ∂M, it must have nonzero x^n-component in any smooth boundary chart. Thus exactly one of the two choices of unit normal has negative x^n-component, which is equivalent to being outward-pointing.

To prove existence, we will show that there exists a smooth outward unit normal field in a neighborhood of each point. By the uniqueness result above, these vector fields all agree where they overlap, so the resulting vector field is globally defined.

Let $p \in \partial M$. By Proposition 11.24, there exists a smooth adapted orthonormal frame $(E_1, \dots, E_n)$ in a neighborhood U of p. In this frame, E_n is a smooth unit normal vector field along ∂M. If we assume (by shrinking U if necessary) that U is connected, then E_n must be either inward-pointing or outward-pointing on all of $\partial M \cap U$. Replacing E_n by $-E_n$ if necessary, we obtain a smooth outward-pointing unit normal vector field defined near p. This completes the proof. □

The next corollary is immediate.

Corollary 13.27. *If (M, g) is an oriented Riemannian manifold with boundary and $\widetilde{g}$ is the induced Riemannian metric on ∂M, then the volume form of $\widetilde{g}$ is*

$$dV_{\widetilde{g}} = (N \lrcorner dV_g)|_{\partial M},$$

where N is the outward unit normal vector field along ∂M.

Problems

13-1. Suppose M is a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Show that M is orientable. Use this to give another proof that $\mathbb{S}^n$ is orientable.

13-2. Suppose $\pi : \widetilde{M} \to M$ is a smooth covering map and M is orientable. Show that $\widetilde{M}$ is also orientable.

13-3. Suppose M and N are oriented smooth manifolds and $F : M \to N$ is a local diffeomorphism. If M is connected, show that F is either orientation-preserving or orientation-reversing.

13-4. Suppose M is a connected, oriented, smooth manifold and Γ is a discrete group acting smoothly, freely, and properly on M. We say