(a) There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.
(b) There exists a continuous map $V: \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$.
(c) The antipodal map $\alpha: \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\operatorname{Id}_{\mathbb{S}^n}$.
(d) The antipodal map $\alpha: \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.
(e) n is odd.

[Hint: Use Problems 8-7, 13-5, and 14-21.]

and, when $\dim M = 3$,

$$\operatorname{curl} X = \left(* \, dX^\flat\right)^\#.$$

14-16. Let (M,g) be a compact, oriented Riemannian n-manifold. For $1 \le k \le n$, define a map $d^*\colon \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ by $d^*\omega = (-1)^{n(k+1)+1} * d * \omega$, where $*$ is the Hodge star operator defined in Problem 14-12. Extend this definition to 0-forms by defining $d^*\omega = 0$ for $\omega \in \mathcal{A}^0(M)$.

(a) Show that $d^* \circ d^* = 0$.

(b) Show that the formula

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle_g \, dV_g$$

defines an inner product on $\mathcal{A}^k(M)$ for each k, where $\langle \cdot, \cdot \rangle_g$ is the pointwise inner product on forms defined in Problem 14-12.

(c) Show that $(d^*\omega, \eta) = (\omega, d\eta)$ for all $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^{k-1}(M)$.

14-17. On $\mathbb{R}^3$ with the Euclidean metric, show that the curl operator we have defined is given by the classical formula:

$$\begin{aligned}\operatorname{curl}&\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right)\\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z}.\end{aligned}$$

14-18. Show that any finite product $M_1 \times \cdots \times M_k$ of smooth manifolds with corners is again a smooth manifold with corners. Give a counterexample to show that a finite product of smooth manifolds with boundary need not be a smooth manifold with boundary.

14-19. Suppose M is a smooth manifold with corners, and let $\mathcal{C}$ denote the set of corner points of M. Show that $M \smallsetminus \mathcal{C}$ is a smooth manifold with boundary.

14-20. Show that the divergence operator on an oriented Riemannian manifold does not depend on the choice of orientation, and conclude that it is invariantly defined on all Riemannian manifolds.

14-21. Let M and N be compact, connected, oriented, smooth manifolds, and suppose $F, G\colon M \to N$ are diffeomorphisms. If F and G are homotopic, show that they are either both orientation-preserving or both orientation-reversing. [Hint: Use the Whitney approximation theorem and Stokes's theorem on $M \times I$.]

14-22. The Hairy Ball Theorem: *There exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.* ("You cannot comb the hair on a ball.") Prove this by showing that the following are equivalent:

14-12. Let (M, g) be an oriented Riemannian n-manifold. This problem outlines an important generalization of the operator $*\colon C^\infty(M) \to \mathcal{A}^n(M)$ defined in this chapter.

(a) For each $k = 1, \dots, n$, show that g determines a unique inner product on $\Lambda^k(T_pM)$ (denoted by $\langle \cdot, \cdot \rangle_g$, just like the inner product on T_pM) satisfying

$$\left\langle \omega^1 \wedge \dots \wedge \omega^k, \eta^1 \wedge \dots \wedge \eta^k \right\rangle_g = \det\left(\left\langle (\omega^i)^\#, (\eta^j)^\# \right\rangle_g \right)$$

whenever $\omega^1, \dots, \omega^k, \eta^1, \dots, \eta^k$ are 1-forms. [Hint: Define the inner product locally by declaring $\{\varepsilon^I|_p : I \text{ is increasing}\}$ to be an orthonormal basis for $\Lambda^k(T_pM)$ whenever (ε^i) is the coframe dual to a local orthonormal frame, and then prove that the resulting inner product is independent of the choice of frame.]

(b) For each $k = 0, \dots, n$, show that there is a unique smooth bundle map $*\colon \Lambda^k M \to \Lambda^{n-k} M$ satisfying

$$\omega \wedge * \eta = \langle \omega, \eta \rangle_g \, dV_g.$$

(For $k = 0$, interpret the inner product as ordinary multiplication.) This map is called the *Hodge star operator.* [Hint: First prove uniqueness, and then define $*$ locally by setting

$$*\left(\varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k}\right) = \pm \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{n-k}}$$

in terms of an orthonormal coframe (ε^i), where the indices $j_1, \dots, j_{n-k}$ are chosen so that $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ is some permutation of $(1, \dots, n)$.]

(c) Show that $*\colon \Lambda^0 M \to \Lambda^n M$ is given by $* f = f \, dV_g$.

(d) Show that $* * \omega = (-1)^{k(n-k)} \omega$ if ω is a k-form.

14-13. Consider $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric and the standard orientation.

(a) Calculate $* dx^i$ for $i = 1, \dots, n$.

(b) Calculate $*(dx^i \wedge dx^j)$ in the case $n = 4$.

14-14. Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be ***self-dual*** if $* \omega = \omega$, and ***anti-self-dual*** if $* \omega = -\omega$.

(a) Show that every 2-form ω on M can be written uniquely as a sum of a self-dual form and an anti-self-dual form.

(b) On $M = \mathbb{R}^4$ with the Euclidean metric, determine the self-dual and anti-self-dual forms in standard coordinates.

14-15. Let (M, g) be an oriented Riemannian manifold and $X \in \mathcal{T}(M)$. Show that

$$X \lrcorner \, dV_g = * X^\flat,$$
$$\operatorname{div} X = * d * X^\flat,$$

Although the definition used here conflicts with the traditional definition of the Laplacian on $\mathbb{R}^n$ (see Problem 14-11), it has two distinct advantages: Our Laplacian has nonnegative eigenvalues (see Problem 14-9), and it agrees with the Laplace–Beltrami operator defined on differential forms (see Problems 15-9 and 15-10). When reading any book or article that mentions the Laplacian, you have to be careful to determine which sign convention the author is using.]

14-9. Let (M, g) be a compact, connected, oriented Riemannian manifold without boundary, and let Δ be its Laplacian. A real number λ is called an *eigenvalue* of Δ if there exists a smooth real-valued function u on M, not identically zero, such that $\Delta u = \lambda u$. In this case, u is called an *eigenfunction* corresponding to λ.

(a) Prove that 0 is an eigenvalue of Δ, and that all other eigenvalues are strictly positive.

(b) If u and v are eigenfunctions corresponding to distinct eigenvalues, show that $\int_M uv\, dV_g = 0$.

14-10. Let M be a compact, connected, oriented Riemannian n-manifold with nonempty boundary. A number $\lambda \in \mathbb{R}$ is called a *Dirichlet eigenvalue* for M if there exists a smooth real-valued function u on M, not identically zero, such that $\Delta u = \lambda u$ and $u|_{\partial M} = 0$. Similarly, λ is called a *Neumann eigenvalue* if there exists such a u satisfying $\Delta u = \lambda u$ and $Nu|_{\partial M} = 0$, where N is the outward unit normal.

(a) Show that every Dirichlet eigenvalue is strictly positive.

(b) Show that 0 is a Neumann eigenvalue, and all other Neumann eigenvalues are strictly positive.

14-11. Let (M, g) be an oriented Riemannian n-manifold with or without boundary.

(a) In any oriented smooth local coordinates (x^i), show that

$$\operatorname{div}\left(X^i \frac{\partial}{\partial x^i}\right) = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(X^i \sqrt{\det g}\right),$$

where $\det g = \det(g_{kl})$ is the determinant of the component matrix of g in these coordinates.

(b) Show that the Laplacian is given in any oriented smooth local coordinates by

$$\Delta u = -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(g^{ij} \sqrt{\det g}\, \frac{\partial u}{\partial x^j}\right).$$

(c) Conclude that on $\mathbb{R}^n$ with the Euclidean metric and standard coordinates,

$$\operatorname{div}\left(X^i \frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{n} \frac{\partial X^i}{\partial x^i}, \qquad \Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{(\partial x^i)^2}.$$

where the integrals on the right-hand side are defined as the limits as $R \nearrow 1$ of the integrals over $\overline{B}_R(0)$. Be sure to justify the limits.

14-5. Suppose $\widetilde{M}$ and M are smooth n-manifolds and $\pi\colon \widetilde{M} \to M$ is a smooth k-sheeted covering map.

(a) If $\widetilde{M}$ and M are oriented and π is orientation-preserving, show that $\int_{\widetilde{M}} \pi^*\omega = k \int_M \omega$ for any compactly supported n-form ω on M.

(b) If μ is any compactly supported density on M, show that $\int_{\widetilde{M}} \pi^*\mu = k \int_M \mu$.

14-6. If M is a compact, smooth, oriented manifold with boundary, show that there does not exist a smooth retraction of M onto its boundary. [Hint: Consider an orientation form on ∂M.]

14-7. Let (M, g) be a compact, oriented Riemannian manifold with boundary, let $\widetilde{g}$ denote the induced Riemannian metric on ∂M, and let N be the outward unit normal vector field along ∂M.

(a) Show that the divergence operator satisfies the following product rule for $f \in C^\infty(M)$, $X \in \mathfrak{T}(M)$:

$$\operatorname{div}(fX) = f \operatorname{div} X + \langle \operatorname{grad} f, X\rangle_g.$$

(b) Prove the following "integration by parts" formula:

$$\int_M \langle \operatorname{grad} f, X\rangle_g \, dV_g = \int_{\partial M} f\langle X, N\rangle_g \, dV_{\widetilde{g}} - \int_M (f \operatorname{div} X)\, dV_g.$$

(c) Explain what this has to do with integration by parts.

14-8. Let (M, g) be an oriented Riemannian manifold with or without boundary. The linear operator $\Delta\colon C^\infty(M) \to C^\infty(M)$ defined by $\Delta u = -\operatorname{div}(\operatorname{grad} u)$ is called the *Laplace operator* or *Laplacian*. A function $u \in C^\infty(M)$ is said to be *harmonic* if $\Delta u = 0$.

(a) If M is compact, prove *Green's identities*:

$$\int_M u\Delta v \, dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g \, dV_g - \int_{\partial M} u\, Nv \, dV_{\widetilde{g}},$$

$$\int_M (u\Delta v - v\Delta u)\, dV_g = \int_{\partial M} (v\, Nu - u\, Nv) dV_{\widetilde{g}},$$

where N and $\widetilde{g}$ are as in Problem 14-7.

(b) If M is compact and connected and $\partial M = \varnothing$, show that the only harmonic functions on M are the constants.

(c) If M is compact and connected, $\partial M \neq \varnothing$, and u, v are harmonic functions on M whose restrictions to ∂M agree, show that $u \equiv v$.

[Remark: There is no general agreement about the sign convention for the Laplacian on a Riemannian manifold, and many authors define the Laplacian to be the negative of the one we have defined here.

Using these facts, together with the divergence theorem on $\widehat{M}$ and the result of Problem 14-5, we compute

$$\begin{aligned}
2\int_M (\operatorname{div} X)\, dV_g &= \int_{\widehat{M}} \widehat{\pi}^* \left((\operatorname{div} X)\, dV_g\right) = \int_{\widehat{M}} \left(\operatorname{div} \widehat{X}\right) |dV_{\widehat{g}}| \\
&= \int_{\widehat{M}} \left(\operatorname{div} \widehat{X}\right) dV_{\widehat{g}} = \int_{\partial\widehat{M}} \langle \widehat{X}, \widehat{N}\rangle_{\widehat{g}}\, dV_{\widetilde{g}} \\
&= \int_{\partial\widehat{M}} \langle \widehat{X}, \widehat{N}\rangle_{\widehat{g}}\, |dV_{\widetilde{g}}| = \int_{\partial\widehat{M}} \left(\widehat{\pi}|_{\partial\widehat{M}}\right)^* \left(\langle X, N\rangle_g\, dV_{\widetilde{g}}\right) \\
&= 2\int_{\partial M} \langle X, N\rangle_g\, dV_{\widetilde{g}}.
\end{aligned}$$

Dividing both sides by 2 yields (14.15). □

Problems

14-1. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ denote the 2-torus, defined by $w^2 + x^2 = y^2 + z^2 = 1$, with the orientation determined by its product structure (see Exercise 13.4). Compute $\int_{\mathbb{T}^2} \omega$, where ω is the following 2-form on $\mathbb{R}^4$:

$$\omega = xyz\, dw \wedge dy.$$

14-2. Let D denote the torus of revolution in $\mathbb{R}^3$ obtained by revolving the circle $(y-2)^2 + z^2 = 1$ around the z-axis (Example 11.23), with its induced Riemannian metric and with the orientation determined by the outward unit normal.

(a) Compute the surface area of D.
(b) Compute the integral over D of the function $f(x,y,z) = z^2 + 1$.
(c) Compute the integral over D of the 2-form $\omega = z\, dx \wedge dy$.

14-3. Let ω be the $(n-1)$-form on $\mathbb{R}^n \smallsetminus \{0\}$ defined by

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x^i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \qquad (14.16)$$

(a) Show that $\omega|_{\mathbb{S}^{n-1}}$ is the Riemannian volume form of $\mathbb{S}^{n-1}$ with respect to the round metric.
(b) Show that ω is closed but not exact on $\mathbb{R}^n \smallsetminus \{0\}$.

14-4. Define maps $F_+, F_- : \mathbb{B}^2 \to \mathbb{S}^2$ by

$$F_\pm(u,v) = \left(u, v, \pm\sqrt{1-u^2-v^2}\right).$$

If ω is a smooth 2-form on $\mathbb{S}^2$, show that

$$\int_{\mathbb{S}^2} \omega = \int_{\mathbb{B}^2} F_+^*\omega - \int_{\mathbb{B}^2} F_-^*\omega,$$