GEOMETRY AND TOPOLOGY PRELIMINARY EXAM
FALL 2020

[1] Let $M_2(\mathbb{R})$ be the set of all real 2×2 matrices. Consider the subset given by $\mathrm{SL}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det A = 1\}$.

(1) Show that $\mathrm{SL}_2(\mathbb{R})$ is a 3-dimensional smooth manifold.

(2) Regarding the tangent space $T_A\mathrm{SL}_2(\mathbb{R})$ at A as a linear subspace of $T_A M_2(\mathbb{R}) \cong M_2(\mathbb{R})$, show that

$$T_A\mathrm{SL}_2(\mathbb{R}) = A \cdot \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \middle| p + s = 0 \right\} \subset M_2(\mathbb{R}).$$

[2] On $\mathbb{R}^3$, consider a distribution generated by the following vector fields:

$$X = \frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}, \qquad Z = -y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}.$$

(1) Show that the above distribution is an involutive 2-dimensional distribution.

(2) Calculate the integral manifold through the origin.

[3] Let α and β be differential forms on a manifold M and let X be a vector field. Prove that

$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta)$$

where L_X denotes the Lie derivative.

[4] Is there an onto submersion $S^3 \to S^2$?

[5] Given a Riemannian manifold (M, g), show that for each $p \in M$, there exists a neighborhood U of p such that every geodesic line in U is a length minimizing curve.

[6] Let S be a compact orientable surface of genus 2. Compute the fundamental group $\pi_1(S)$. (Hint: What is the fundamental group of a punctured torus?)

[7] Let $n \geq 0$. Prove that every continuous map $f : D^n \to D^n$ from the closed n-disk to itself has a fixed point.