FINAL HOMEWORKS.
Due Monday March 15.

GP: 4.9: 8 , 10

Flanders 4.7, p 48 , 4, 5, 6, 7

A problem on Divergence.

a) If μ is a volume form on a manifold Q and X is a vector field on the same manifold show that $L_X \mu = f\mu$ for some function f. [Hint: if Q has dimension n, for what integer k is μ a k-form? $L_X \mu$?]

b) If the manifold is $\mathbf{R}^n$ and μ is the standard volume form show that the function f from part (a) is $f = div(X)$, the usual divergence of a vector field in $\mathbf{R}^n$ when computed in orthonormal linear coordinates.

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Suggested order. Do GP 4.9, number 8 before starting in on Flanders.

Hints. For GP 4.9, number 8, and Flanders 4.9, numbers 4, 5 and 6 it will be more efficient to use the normal vector ["theorem egregium"] way of computing curvature, available for smoothly embedded surfaces $\Sigma \subset \mathbf{R}^3$ which I now summarize.

How to compute curvature using the normal vector field.

Let $N : \Sigma \to S^2$ be the unit normal to the surface. Then $dN : T_p\Sigma \to T_{N(p)}S^2$. But $T_{N(p)}S^2 = (N(p))^\perp = T_p\Sigma$ so that $dN(p)$ is a linear map from a two-dimensional vector space to itself. As such, its determinant and trace are well-defined, independent of basis.

Theorem. $K(p) = det(dN(p))$ is the Gauss curvature of Σ at $p \in \Sigma$, which depends only on the 'first fundamental form $ds^2|_\Sigma$.

Flanders asks you to compute "the curvatures". The 'other' curvature he refers to is the the mean curvature H, given by

$$H(p) = (1/2)tr(dN(p)).$$

The theorem is called the Theorem Egregium, 'remarkable theorem' in Latin. What makes in remarkable is that the way the normal vector changes as we move about the surfac clearly depends on how we have embedded the surface in space. It depends on the ambient geometry. However, the Gauss curvature is "intrinsic" : it only depends on the induced geometry ("first fundamental form").