Homework 6: Lie groups.

1. The Lie group $O(n)$, as a set, is the solution space to the quadratic matrix equation

$$AA^t = Id.$$

within the space $gl(n, \mathbb{R}) = End(\mathbb{V})$ all real n by n matrices. It may help to think of $gl(n) = End(\mathbb{V})$ where $\mathbb{V} = \mathbb{R}^n$. The left hand side of this equation defines a map $F(A) = AA^t$ from the real vector space $End(\mathbb{V})$ to the real vector space $Sym(\mathbb{V})$ of symmetric operators (or matrices) on $\mathbb{V}$.

a) Compute the derivative of F at a matrix $A_0 \in End(\mathbb{V})$ using the method of curves. Your formula should be one for a linear operator $B \mapsto dF_{A_0}(B)$.

b) Show that the identity, Id, is a regular value for F.

c) From (b) conclude that $O(n)$ is a manifold.

d) Compute the dimension of $O(n)$ and describe its tangent space at $A = I$.

e) Show that $O(n)$ is a Lie group.

2. Repeat problem 1 for $U(n)$ which is defined by $AA^* = Id$, where $A \subset gl(n, \mathbb{C})$ the space of all n by n complex matrices and where A^* is its hermitian conjugate: the matrix whose entries are the conjugates of the transpose of A. Begin by identifying the right target vector space for $A \mapsto AA^*$.

3. Repeat problem 1 for the symplectic group $Sp(n)$. To define $Sp(n)$ choose a symplectic form ω for a real vector space $\mathbb{V}$ of dimension $2n$. Then $Sp(n) = Sp(\omega) \subset GL(2n) = Gl(\mathbb{V})$ is defined by the equation

$$\omega(Av, Aw) = \omega(v, w) \text{ for all } v, w \in \mathbb{V}.$$

You will need to do a 'step 0" in order to place the definition of $Sp(n)$ into the context of problem 1. **Step 0.** The Darboux theorem in its simplest form asserts that $\mathbb{V}$ admits linear 'Darboux' coordinates $q^1, \dots, q^n, p_1, \dots p_n$ such that $\omega = \Sigma_{i=1}^n dp_i \wedge dq^i$. Now define a linear operator J by

$$\omega(v, w) = \langle v, Jw \rangle$$

where the inner product $\langle \cdot, \cdot \rangle$ is the one for which the Darboux coordinates are orthonormal. Verify that J has the q, p block form

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

Use J now to define a quadratic equation for $A \in Sp(n)$ similar to that defining $O(n)$. Now continue with the rest of the steps. (Warning: there are two different Lie groups both commonly denoted "$Sp(n)$". I am using the one that does not involve quaternions!)

4. Look up what it means for a manifold to be 'parallelizable'. Prove that every connected Lie group is parallelizable.

5. The quaternions $\mathbb{H}$ are a real 4-dimensional algebra with basis $1, i, j, k$. Look up the definition of the quaternions. Look up the definition of quaternionic conjugation $q \mapsto \bar{q}$.

a) Prove that the group of unit quaternions: $\{q : q\bar{q} = 1\}$ forms a Lie group under quaternionic multiplication and that as a manifold it is diffeomorphic to the 3-sphere S^3. This group is typically denoted "$Sp(1)$ - so you are see the other "Sp" now.

b) Show that the Lie algebra $sp(1)$ of the group of unit quaternions is the space of purely imaginary quaternions: $\{h : \bar{h} = -h\}$ Figure out its Lie bracket.

c) $Sp(1)$ acts on $\mathbb{R}^3 \cong Im(\mathbb{H})$ by conjugation: $(q, h) \mapsto qh\bar{q}$. Show that this defines a smooth homomorphism $Sp(1) \to SO(3)$ which is onto. Find its kernel. Hint: the standard inner product on $\mathbb{R}^3$ is induced by $\langle h_1, h_2 \rangle = Re(h_1 \bar{h}_2)$.

d) $Sp(1) \times Sp(1)$ acts on $\mathbb{R}^4 \cong \mathbb{H}$ by $((g_1, g_2), q) \mapsto g_1 q \bar{g}_2$. Show that this action defines a homomorphism $Sp(1) \times Sp(1) \to SO(4)$ which is onto. Find its kernel.

e) Look up the definition of "$SU(2)$". Show that $SU(2)$ is isomorphic as a LIe group to $Sp(1)$.